

Minimal unitary representation of $SO^*(8) = SO(6, 2)$ and its $SU(2)$ deformations as massless $6D$ conformal fields and their supersymmetric extensions

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ABSTRACT: We study the minimal unitary representation (minrep) of $SO(6, 2)$ over an Hilbert space of functions of five variables, obtained by quantizing its quasiconformal realization. The minrep of $SO(6, 2)$, which coincides with the minrep of $SO^*(8)$ similarly constructed, corresponds to a massless conformal scalar field in six spacetime dimensions. There exists a family of “deformations” of the minrep of $SO^*(8)$ labeled by the spin t of an $SU(2)_T$ subgroup of the little group $SO(4)$ of lightlike vectors. These deformations labeled by t are positive energy unitary irreducible representations of $SO^*(8)$ that describe massless conformal fields in six dimensions. The $SU(2)_T$ spin t is the six dimensional counterpart of $U(1)$ deformations of the minrep of $4D$ conformal group $SU(2, 2)$ labeled by helicity. We also construct the supersymmetric extensions of the minimal unitary representation of $SO^*(8)$ to minimal unitary representations of $OSp(8^*|2N)$ that describe massless six dimensional conformal supermultiplets. The minimal unitary supermultiplet of $OSp(8^*|4)$ is the massless supermultiplet of $(2, 0)$ conformal field theory that is believed to be dual to M-theory on $AdS_7 \times S^4$.

KEYWORDS: AdS/CFT, Minimal Unitary Representations, Conformal Group.

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1. Introduction

Unitary representations of noncompact U-duality groups of extended supergravity theories were first studied in early 1980s [1–3], motivated by the idea that, in a quantum theory, global symmetries must be realized unitarily, as well as by attempts to derive a composite grand unified theory (GUT) from $N = 8$ supergravity [4–6]. In the composite model of [4], the local R-symmetry group $SU(8)$ of $N = 8$ supergravity was conjectured to become dynamical at the quantum level. A similar scenario based on the exceptional supergravity theory [7] leads to E_6 GUT with a family group $U(1)$. After the discovery of counter terms at higher loops in $N = 8$ supergravity and the Green-Schwarz anomaly cancellation in superstring theory [8], the work on composite models was all but abandoned. Recent work proving cancellations of divergences in $N = 8$ supergravity up to four loops [9–18] revived the question of finiteness of $N = 8$ supergravity as well as of exceptional supergravity.

The oscillator method developed in [2], to construct the relevant unitary representations of noncompact U-duality groups of supergravity theories, generalized and unified previous special constructions in the physics literature. The formulation of [2] was later extended to noncompact supergroups in [19] using bosonic as well as fermionic oscillators. In these generalized formulations of [2] and [19], one realizes the generators of noncompact groups or supergroups as bilinears of an arbitrary number P (“colors”) of sets of oscillators transforming in an irreducible representation (typically fundamental) of their maximal compact subgroups or subsupergroups. For symplectic groups $Sp(2n, \mathbb{R})$ the minimum value of P turns out to be one, and the resulting unitary representations are simply the singleton representations, which are known as metaplectic representations in the mathematics literature. In general the minimum allowed value of P for a noncompact group is two, and the resulting unitary representations were later referred to as doubleton representations. For example, for the

groups $SU(n, m)$ and $SO^*(2n)$, with maximal compact subgroups $SU(m) \times SU(n) \times U(1)$ and $U(n)$, respectively, one finds that $P_{min} = 2$. Symplectic groups $Sp(2n, \mathbb{R})$ admit only two singleton irreducible representations (irreps). Noncompact groups or supergroups that do not admit singleton representations have an infinite number of doubleton irreps. Since the generators are realized as bilinears of free bosonic and fermionic oscillators, tensoring of the resulting representations is very straightforward within the oscillator approach. Furthermore the oscillator method is simple and yet very powerful for constructing positive energy unitary representations. Even though the positive energy singleton or doubleton irreps do not belong to the discrete series, by tensoring them one obtains positive energy unitary representations that belong, in general, to the holomorphic discrete series representations of the respective noncompact group or supergroup.

The oscillator methods for constructing positive energy unitary representations of noncompact groups and supergroups were applied to spacetime supergroups beginning in the 1980s. The Kaluza-Klein spectrum of IIB supergravity spontaneously compactified over the product space $AdS_5 \times S^5$ was first obtained via the oscillator method by simple tensoring of the CPT self-conjugate doubleton supermultiplet of $N = 8$ AdS_5 superalgebra $PSU(2, 2|4)$ repeatedly with itself and restricting to the CPT self-conjugate short supermultiplets of $PSU(2, 2|4)$ [20]. The CPT self-conjugate doubleton supermultiplet does not have a Poincaré limit in five dimensions and decouples from the Kaluza-Klein spectrum as gauge modes. This led the authors of [20] to the proposal that the field theory of CPT self-conjugate doubleton supermultiplet of $PSU(2, 2|4)$ lives on the boundary of AdS_5 , which can be identified with 4D Minkowski space on which $SO(4, 2)$ acts as a conformal group. Furthermore they pointed out that the unique candidate for this theory is the four dimensional $N = 4$ super Yang-Mills theory that was known to be conformally invariant.

The spectra of the spontaneous compactifications of eleven dimensional supergravity over $AdS_4 \times S^7$ and $AdS_7 \times S^4$, that had been obtained by other methods previously, were fitted into supermultiplets of the symmetry superalgebras $OSp(8|4, \mathbb{R})$ and $OSp(8^*|4)$ obtained by oscillator methods in [21] and [22], respectively. Furthermore, the entire Kaluza-Klein spectra of eleven dimensional supergravity over these two spaces were obtained by tensoring the singleton and scalar doubleton supermultiplets of $OSp(8|4, \mathbb{R})$ and $OSp(8^*|4)$, respectively. The singleton and doubleton supermultiplets themselves do not have a Poincaré limit in four and seven dimensions and decouple from the respective spectra as gauge modes. Again it was proposed that the field theories of the singleton and scalar doubleton supermultiplets live on the boundaries of AdS_4 and AdS_7 as super conformally invariant theories [21, 22].

The importance of these results was not fully realized until the work of Maldacena [23] and subsequent works of Witten [24] and of Gubser et al. [25] and have since become an integral part of the work on AdS/CFT dualities in M/superstring theory which has seen an exponential growth for over more than a decade now.

Noncompact groups were also introduced into physics as spectrum generating symmetry groups during the 1960s. Inspired by the work of physicists on spectrum generating symmetry groups, Joseph introduced the concept of minimal unitary realizations of Lie groups in [26].

These are unitary representations of corresponding noncompact groups over Hilbert spaces of functions of smallest possible (minimal) number of variables. Joseph gave the minimal realizations of the complex forms of classical Lie algebras and of the exceptional Lie algebra \mathfrak{g}_2 in a Cartan-Weil basis. The minimal unitary representation of the split exceptional group $E_{8(8)}$ was first identified within Langland's classification by Vogan [27]. In an important paper, Kostant studied the minimal unitary representation of $SO(4, 4)$ and its relation to triality in [28]. A general study of minimal unitary representations of simply laced groups was given by Kazhdan and Savin [29] and by Brylinski and Kostant [30, 31].

The minimal unitary representations of quaternionic real forms of exceptional Lie groups were studied by Gross and Wallach [32] and those of $SO(p, q)$ in [33–36]. Pioline, Kazhdan and Waldron [37] reformulated the minimal unitary representations of simply laced groups given in [29] and gave the spherical vectors for the simply laced exceptional groups necessary for the construction of modular forms. The relation of minimal representations of $SO(p, q)$ to conformal geometry was studied rather recently in [38].

Over the last decade, a great deal of progress was made towards the goal of constructing physically relevant unitary representations of U-duality groups of extended supergravity theories. An additional motivation towards this goal was provided by the proposals that certain extensions of U-duality groups may act as spectrum generating symmetry groups of these theories. Work on orbits of extremal black hole solutions in $N = 8$ supergravity and $N = 2$ Maxwell-Einstein supergravity theories with symmetric scalar manifolds led to the proposal that four dimensional U-duality groups act as spectrum generating conformal symmetry groups of the corresponding five dimensional supergravity theories [39–44]. In attempts to find the corresponding spectrum generating symmetry groups of extremal black hole solutions of four dimensional supergravity theories with symmetric scalar manifolds, geometric quasiconformal realizations of three dimensional U-duality groups were discovered in [40]. Based on this novel geometric realization, quasiconformal extensions of four dimensional U-duality groups were proposed as spectrum generating symmetry groups of the corresponding supergravity theories with symmetric scalar manifolds [40–44]. A concrete implementation of the proposal that three dimensional U-duality groups act as spectrum generating quasiconformal groups was given in [45–47] using the equivalence of equations of attractor flows of spherically symmetric stationary BPS black holes of four dimensional supergravity theories and the geodesic equations of a fiducial particle moving in the target space of the three dimensional supergravity theories obtained by reduction of the $4D$ theories on a timelike circle [48].

Quasiconformal realization of three dimensional U-duality group $E_{8(8)}$ of maximal supergravity in three dimensions is the first known geometric realization of $E_{8(8)}$ [40]. Quasiconformal action of $E_{8(8)}$ leaves invariant a generalized light-cone with respect to a quartic distance function in 57 dimensions. Quasiconformal realizations exist for various real forms of all noncompact groups as well as for their complex forms [40, 49]. Remarkably, the quantization of geometric quasiconformal action of a noncompact group leads directly to its minimal unitary representation as was first shown explicitly for the maximally split exceptional group $E_{8(8)}$ with the maximal compact subgroup $SO(16)$ [50].

The minimal unitary representation of three dimensional U-duality group $E_{8(-24)}$ of the exceptional supergravity [7] was given in [51].

The minimal unitary representations of U-duality groups $F_{4(4)}$, $E_{6(2)}$, $E_{7(-5)}$, $E_{8(-24)}$ and $SO(d+2, 4)$ of $N = 2$ Maxwell-Einstein supergravity theories with symmetric scalar manifolds were studied in [49, 51]. In [52], a unified formulation of the minimal unitary representations of certain noncompact real forms of groups of type A_2 , G_2 , D_4 , F_4 , E_6 , E_7 , E_8 and C_n was given. The minimal unitary representations of $Sp(2n, \mathbb{R})$ are simply the singleton representations. In [52], minimal unitary representations of noncompact groups $SU(m, n)$, $SO(m, n)$, $SO^*(2n)$ and $SL(m, \mathbb{R})$ obtained by quasiconformal methods were also given explicitly. Furthermore, this unified approach was generalized to define and construct the corresponding minimal representations of non-compact supergroups G whose even subgroups are of the form $H \times SL(2, \mathbb{R})$ with H compact. The unified construction with H simple or Abelian leads to the minimal unitary representations of supergroups $G(3)$, $F(4)$ and $OSp(n|2, \mathbb{R})$. The minimal unitary representations of $OSp(n|2, \mathbb{R})$ with even subgroups $SO(n) \times Sp(2, \mathbb{R})$ are the singleton supermultiplets. The minimal realization of the one parameter family of Lie superalgebras $D(2, 1; \sigma)$ with even subgroup $SU(2) \times SU(2) \times SU(1, 1)$ was also presented in [52].

In mathematics literature, the term minimal unitary representation refers, in general, to a unique representation of the respective noncompact group. The symplectic group $Sp(2N, \mathbb{R})$ admits two singleton irreps whose quadratic Casimirs take on the same value. Both of these singleton representations are minimal unitary representations, even though in some of the mathematics literature only the scalar singleton is referred to as the minrep. Similarly one finds that the supergroups $OSp(M|2N, \mathbb{R})$ with the even subgroup $SO(M) \times Sp(2N, \mathbb{R})$ admit two inequivalent singleton supermultiplets [21, 53, 54]. For noncompact groups or supergroups that admit only doubleton irreps, this raises the question as to whether any of the doubleton unitary representations can be identified with the minimal representation, and if so, how the infinite set of doubletons are related to the minrep. More recently, we investigated this issue for 5D anti-de Sitter or 4D conformal group $SU(2, 2)$ and corresponding supergroups $SU(2, 2|N)$ [55]. We gave a detailed study of the minimal unitary representation of the group $SU(2, 2)$ by quantization of its quasiconformal realization and showed that it coincides with the scalar doubleton representation corresponding to a massless scalar field in four dimensions. Furthermore we showed that the minrep of $SU(2, 2)$ admits a one-parameter family (ζ) of deformations, and for a positive (negative) integer value of the deformation parameter ζ , one obtains a positive energy unitary irreducible representation of $SU(2, 2)$ corresponding to a massless conformal field in four dimensions transforming in $\left(0, \frac{\zeta}{2}\right) \left(\left(-\frac{\zeta}{2}, 0\right)\right)$ representation of the Lorentz subgroup, $SL(2, \mathbb{C})$ of $SU(2, 2)$. These are simply the doubleton representations of $SU(2, 2)$ that describe massless conformal fields in four dimensions [56, 57]. They were referred to as ladder (or most degenerate discrete series) unitary representations by Mack and Todorov, who showed that they remain irreducible under restriction to the Poincaré subgroup [58]. Hence the deformation parameter can be identified with twice the helicity

\hbar of the corresponding massless representation of the Poincaré group. We extended these results to the minimal unitary representations of supergroups $SU(2, 2|N)$ with the even subgroup $SU(2, 2) \times U(N)$ and their deformations. The minimal unitary supermultiplet of $SU(2, 2|N)$ coincides with the CPT self-conjugate (scalar) doubleton supermultiplet, and for $PSU(2, 2|4)$ it is simply the four dimensional $N = 4$ Yang-Mills supermultiplet. Again in the supersymmetric case, one finds a one-parameter family of deformations of the minimal unitary supermultiplet of $SU(2, 2|N)$. Each integer value of the deformation parameter ζ leads to a unique unitary supermultiplet of $SU(2, 2|N)$. The minimal unitary supermultiplet of $SU(2, 2|N)$ and its deformations turn out to be precisely the doubleton supermultiplets that were constructed and studied using the oscillator method earlier [20, 56, 57]. These results extend to the minreps of $SU(m, n)$ and of $SU(m, n|N)$ and their deformations in a straightforward manner.

In this paper we give a detailed study of the minimal unitary representation of $7D$ anti-de Sitter or $6D$ conformal group $SO^*(8) = SO(6, 2)$, obtained by quantizing its realization as a quasiconformal group that leaves invariant a quartic light-cone in nine dimensions, its deformations and their supersymmetric extensions to supermultiplets of $OSp(8^*|2N)$. The oscillator construction of the positive energy unitary supermultiplets of $OSp(8^*|2N)$ were first given in [22]. These unitary supermultiplets were further studied in [59, 60] where it was shown that the doubleton supermultiplets correspond to massless conformal supermultiplets in six dimensions. A classification of the positive energy unitary supermultiplets of $6D$ superconformal algebras using other methods was given in [61, 62]. The oscillator construction of positive energy representations of general supergroups $OSp(2M^*|2N)$ with maximal compact subgroup $SO^*(2M) \times USp(2N)$ was given in [63].

The plan of our paper is as follows. In section 2 we review the geometric quasiconformal realizations of groups $SO(d+2, 2)$ as invariance groups of a light-cone with respect to a quartic distance function in $(2d+1)$ dimensional space. The quantization of this geometric realization leads to the minimal unitary representation of $SO(d+2, 2)$ over an Hilbert space of functions in $d+1$ variables. We then specialize and study the case of $SO(6, 2)$ in great detail in section 3. In section 4, we study the minimal unitary realization of $SO^*(8)$ which is isomorphic to $SO(6, 2)$. The transformations relating the $SO^*(8)$ basis to that of $SO(6, 2)$ is given in Appendix C. Section 5 discusses the properties of a distinguished $SU(1, 1)$ subgroup of $SO^*(8)$ generated by singular (isotonic) oscillators. We then give the K-type decomposition¹ of the minrep of $SO^*(8)$ in section 6 and show that it coincides with the K-type decomposition of scalar doubleton representation corresponding to a massless conformal scalar field in six dimensions that were studied in [22, 59, 60]. Section 7 reviews the fermionic construction of relevant representations of $USp(2N)$.

In section 8 and section 9, we give the minimal unitary realization of the superalgebra $OSp(8^*|2N)$ with even subgroup $SO^*(8) \times USp(2N)$ obtained from quantizing its quasiconformal realization. Section 10 presents the minimal unitary supermultiplets of $OSp(8^*|2N)$.

¹K-type decomposition is the decomposition with respect to the maximal compact subgroup.

We devote a special subsection to the minimal unitary supermultiplet of $OSp(8^*|4)$, which is the symmetry supergroup of M-theory compactified over $AdS_7 \times S^4$ and show that it coincides with the $(2, 0)$ doubleton supermultiplet studied in [22, 59, 60]. M-theory compactified over $AdS_7 \times S^4$ is believed to be dual to a $(2, 0)$ six dimensional superconformal theory based on this supermultiplet.

We then study the general deformations of the minrep of $SO^*(8)$, independently of supersymmetry, in section 11, and show that there exist an infinite family of deformations labeled by the spin t of an $SU(2)_{\hat{T}}$ subgroup of the semi-simple part of the little group $SO(4)$ of massless states in six dimensions. For every spin value t , one obtains a positive energy unitary irreducible representation of $SO^*(8)$ corresponding to a massless conformal field in six dimensions with Dynkin labels $(2t, 0, 0)$ with respect to the covering group $SU^*(4)$ of the six dimensional Lorentz group $SO(5, 1)$. The $SU(2)$ spin label t for deformations is the $6D$ analog of the helicity label for deformations of the minrep of $4D$ conformal group [55].

2. Quasiconformal Realizations of $SO(d+2, 2)$ and Their Minimal Unitary Representations

2.1 Geometric realizations of $SO(d+2, 2)$ as quasiconformal groups

Lie algebra of the $(d+2)$ dimensional conformal group $SO(d+2, 2)$ can be given a 5-graded decomposition with respect to its subalgebra $\mathfrak{so}(d) \oplus \mathfrak{so}(1, 1)$ [49]

$$\mathfrak{so}(d+2, 2) = \mathbf{1}^{(-2)} \oplus (\mathbf{d}, \mathbf{2})^{(-1)} \oplus [\Delta \oplus \mathfrak{sp}(2, \mathbb{R}) \oplus \mathfrak{so}(d)] \oplus (\mathbf{d}, \mathbf{2})^{(+1)} \oplus \mathbf{1}^{(+2)} \quad (2.1)$$

where Δ is the $SO(1, 1)$ generator that determines the five grading. The superscript m labels the grade of a generator:

$$[\Delta, \mathfrak{g}^{(m)}] = m \mathfrak{g}^{(m)} \quad (2.2)$$

In the above decomposition, $(\mathbf{d}, \mathbf{2})^{(m)}$ labels the generators transforming in the $(d, 2)$ representation of $SO(d) \times Sp(2, \mathbb{R})$ with grade m . Generators of quasiconformal action are realized as differential operators acting on a $(2d+1)$ dimensional space \mathcal{T} corresponding to the Heisenberg subalgebra generated by the elements of $\mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)}$ subspace. We shall denote the coordinates of the space \mathcal{T} as $\mathcal{X} = (X^{i,\alpha}, x)$, where $X^{i,\alpha}$ transform in the $(d, 2)$ representation of $SO(d) \times Sp(2, \mathbb{R})$, with $i = 1, 2, \dots, d$ and $\alpha = 1, 2$, and x is a singlet coordinate.

Let $\epsilon_{\alpha\beta}$ be the symplectic metric of $Sp(2, \mathbb{R})$ and η_{ij} the $SO(d)$ invariant metric ($\eta_{ij} = -\delta_{ij}$). Then the quartic polynomial in $X^{i,\alpha}$

$$\mathcal{I}_4(X) = \eta_{ij}\eta_{kl}\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}X^{i,\alpha}X^{j,\beta}X^{k,\gamma}X^{l,\delta} \quad (2.3)$$

is invariant under $SO(d) \times Sp(2, \mathbb{R})$ subgroup.

We shall label the generators belonging to various grade subspaces as follows

$$\mathfrak{so}(d+2, 2) = K_- \oplus U_{i,\alpha} \oplus [\Delta \oplus J_{\alpha\beta} \oplus M_{ij}] \oplus \tilde{U}_{i,\alpha} \oplus K_+ \quad (2.4)$$

where $J_{\alpha\beta}$ and M_{ij} are the generators of $Sp(2, \mathbb{R})$ and $SO(d)$ subgroups, respectively. The infinitesimal generators of the quasiconformal action of $SO(d+2, 2)$ take on the form

$$\begin{aligned}
K_+ &= \frac{1}{2} (2x^2 - \mathcal{I}_4) \frac{\partial}{\partial x} - \frac{1}{4} \frac{\partial \mathcal{I}_4}{\partial X^{i,\alpha}} \eta^{ij} \epsilon^{\alpha\beta} \frac{\partial}{\partial X^{j,\beta}} + x X^{i,\alpha} \frac{\partial}{\partial X^{i,\alpha}} \\
U_{i,\alpha} &= \frac{\partial}{\partial X^{i,\alpha}} - \eta_{ij} \epsilon_{\alpha\beta} X^{j,\beta} \frac{\partial}{\partial x} \\
M_{ij} &= \eta_{ik} X^{k,\alpha} \frac{\partial}{\partial X^{j,\alpha}} - \eta_{jk} X^{k,\alpha} \frac{\partial}{\partial X^{i,\alpha}} \\
J_{\alpha\beta} &= \epsilon_{\alpha\gamma} X^{i,\gamma} \frac{\partial}{\partial X^{i,\beta}} + \epsilon_{\beta\gamma} X^{i,\gamma} \frac{\partial}{\partial X^{i,\alpha}} \\
K_- &= \frac{\partial}{\partial x} \\
\Delta &= 2x \frac{\partial}{\partial x} + X^{i,\alpha} \frac{\partial}{\partial X^{i,\alpha}} \\
\tilde{U}_{i,\alpha} &= [U_{i,\alpha}, K_+]
\end{aligned} \tag{2.5}$$

where $\epsilon^{\alpha\beta}$ is the inverse symplectic metric such that $\epsilon^{\alpha\beta} \epsilon_{\beta\gamma} = \delta^\alpha_\gamma$. Using

$$\frac{\partial \mathcal{I}_4}{\partial X^{i,\alpha}} = -4 \eta_{ij} \eta_{kl} X^{j,\beta} X^{k,\gamma} X^{l,\delta} \epsilon_{\beta\gamma} \epsilon_{\alpha\delta} \tag{2.6}$$

one obtains the explicit form of grade +1 generators $\tilde{U}^{i,\alpha}$:

$$\begin{aligned}
\tilde{U}_{i,\alpha} &= \eta_{ij} \epsilon_{\alpha\delta} \left(\eta_{kl} \epsilon_{\beta\gamma} X^{j,\beta} X^{k,\gamma} X^{l,\delta} - x X^{j,\delta} \right) \frac{\partial}{\partial x} + x \frac{\partial}{\partial X^{i,\alpha}} \\
&\quad - \eta_{ij} \epsilon_{\alpha\beta} X^{j,\beta} X^{k,\gamma} \frac{\partial}{\partial X^{k,\gamma}} - \eta_{jk} \epsilon_{\alpha\delta} X^{k,\delta} X^{j,\gamma} \frac{\partial}{\partial X^{i,\gamma}} \\
&\quad + \eta_{ij} \epsilon_{\alpha\gamma} X^{k,\gamma} X^{j,\beta} \frac{\partial}{\partial X^{k,\beta}} + \eta_{ij} \epsilon_{\beta\gamma} X^{j,\beta} X^{k,\gamma} \frac{\partial}{\partial X^{k,\alpha}}
\end{aligned} \tag{2.7}$$

The above generators satisfy the following commutation relations:

$$\begin{aligned}
[M_{ij}, M_{kl}] &= \eta_{jk} M_{il} - \eta_{ik} M_{jl} - \eta_{jl} M_{ik} + \eta_{il} M_{jk} \\
[J_{\alpha\beta}, J_{\gamma\delta}] &= \epsilon_{\gamma\beta} J_{\alpha\delta} + \epsilon_{\gamma\alpha} J_{\beta\delta} + \epsilon_{\delta\beta} J_{\alpha\gamma} + \epsilon_{\delta\alpha} J_{\beta\gamma}
\end{aligned} \tag{2.8a}$$

$$\begin{aligned}
[\Delta, K_\pm] &= \pm 2 K_\pm & [K_-, K_+] &= \Delta \\
[\Delta, U_{i,\alpha}] &= -U_{i,\alpha} & [\Delta, \tilde{U}_{i,\alpha}] &= \tilde{U}_{i,\alpha} \\
[U_{i,\alpha}, K_+] &= \tilde{U}_{i,\alpha} & [\tilde{U}_{i,\alpha}, K_-] &= -U_{i,\alpha} \\
[U_{i,\alpha}, U_{j,\beta}] &= 2 \eta_{ij} \epsilon_{\alpha\beta} K_- & [\tilde{U}_{i,\alpha}, \tilde{U}_{j,\beta}] &= 2 \eta_{ij} \epsilon_{\alpha\beta} K_+
\end{aligned} \tag{2.8b}$$

$$\begin{aligned}
[M_{ij}, U_{k,\alpha}] &= \eta_{jk} U_{i,\alpha} - \eta_{ik} U_{j,\alpha} & [M_{ij}, \tilde{U}_{k,\alpha}] &= \eta_{jk} \tilde{U}_{i,\alpha} - \eta_{ik} \tilde{U}_{j,\alpha} \\
[J_{\alpha\beta}, U_{i,\gamma}] &= \epsilon_{\gamma\beta} U_{i,\alpha} + \epsilon_{\gamma\alpha} U_{i,\beta} & [J_{\alpha\beta}, \tilde{U}_{i,\gamma}] &= \epsilon_{\gamma\beta} \tilde{U}_{i,\alpha} + \epsilon_{\gamma\alpha} \tilde{U}_{i,\beta}
\end{aligned} \tag{2.8c}$$

$$\left[U_{i,\alpha}, \tilde{U}_{j,\beta} \right] = \eta_{ij} \epsilon_{\alpha\beta} \Delta - 2 \epsilon_{\alpha\beta} M_{ij} - \eta_{ij} J_{\alpha\beta} \quad (2.8d)$$

One defines the quartic norm $\mathcal{N}_4(\mathcal{X})$ of a vector $\mathcal{X} = (X^{i,\alpha}, x)$ in \mathcal{T} as

$$\mathcal{N}_4(\mathcal{X}) := \mathcal{I}_4(X) + 2x^2 \quad (2.9)$$

and the “quartic distance” between any two points with coordinate vectors \mathcal{X} and \mathcal{Y} as

$$d(\mathcal{X}, \mathcal{Y}) := \mathcal{N}_4(\delta(\mathcal{X}, \mathcal{Y})) \quad (2.10)$$

where $\delta(\mathcal{X}, \mathcal{Y})$ is the “symplectic” difference of two vectors \mathcal{X} and \mathcal{Y} in the $(2d+1)$ dimensional space \mathcal{T} given by [40, 49]

$$\delta(\mathcal{X}, \mathcal{Y}) := \left(X^{i,\alpha} - Y^{i,\alpha}, x - y - \eta_{ij} \epsilon_{\alpha\beta} X^{i,\alpha} Y^{j,\beta} \right). \quad (2.11)$$

Under the quasiconformal action of the generators of $SO(d+2, 2)$ the quartic distance function transforms as:

$$\begin{aligned} \Delta d(\mathcal{X}, \mathcal{Y}) &= 4 d(\mathcal{X}, \mathcal{Y}) \\ \tilde{U}_{i,\alpha} d(\mathcal{X}, \mathcal{Y}) &= -2 \eta_{ij} \epsilon_{\alpha\beta} \left(X^{j,\beta} + Y^{j,\beta} \right) d(\mathcal{X}, \mathcal{Y}) \\ K_+ d(\mathcal{X}, \mathcal{Y}) &= 2 (x + y) d(\mathcal{X}, \mathcal{Y}) \\ M_{ij} d(\mathcal{X}, \mathcal{Y}) &= 0 \\ J_{\alpha\beta} d(\mathcal{X}, \mathcal{Y}) &= 0 \\ U_{i,\alpha} d(\mathcal{X}, \mathcal{Y}) &= 0 \\ K_- d(\mathcal{X}, \mathcal{Y}) &= 0 \end{aligned} \quad (2.12)$$

They imply that light-like separations

$$d(\mathcal{X}, \mathcal{Y}) = 0$$

are left invariant under the quasiconformal action. In other words, the quasiconformal action of $SO(d+2, 2)$ leaves the “light-cone” in \mathcal{T} with respect to the *quartic* distance function invariant.

2.2 Minimal unitary representations of $SO(d+2, 2)$ from quantization of their quasiconformal realizations

Minimal unitary representations of noncompact groups can be obtained by the quantization of their geometric realizations as quasiconformal groups [47, 49–52]. In this section we shall review the minimal unitary representations of orthogonal groups $SO(d+2, 2)$ thus obtained following [49, 52]. Let X^i and P_i be the quantum mechanical coordinate and momentum operators on $\mathbb{R}^{(d)}$ satisfying the canonical commutation relations

$$[X^i, P_j] = i \delta_j^i. \quad (2.13)$$

The generators belonging to the subspace $\mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)}$ of the Lie algebra of $SO(d+2, 2)$ form an Heisenberg algebra

$$[U_{i,\alpha}, U_{j,\beta}] = 2\eta_{ij}\epsilon_{\alpha\beta} K_- \quad (2.14)$$

with K_- playing the role of the central charge. We shall relabel the generators and define

$$U_{i,1} \equiv U_i \quad U_{i,2} \equiv V_i \quad (2.15)$$

and realize the Heisenberg algebra (equation (2.14)) in terms of coordinate and momentum operators X^i , P_i and an extra “central charge coordinate” x as

$$\begin{aligned} U_i &= xP_i & V^i &= xX^i \\ K_- &= \frac{1}{2}x^2 \end{aligned} \quad (2.16)$$

$$[V^i, U_j] = 2i\delta_j^i K_- \quad (2.17)$$

By introducing the quantum mechanical momentum operator p , conjugate to the central charge coordinate x , such that

$$[x, p] = i \quad (2.18)$$

one can realize the generators of $SO(d) \times Sp(2, \mathbb{R}) \times SO(1, 1)$ subgroup belonging to the grade zero subalgebra of $\mathfrak{so}(d+2, 2)$ as bilinears of canonically conjugate pairs of coordinate and momentum operators [49, 52]:

$$\begin{aligned} M_{ij} &= -i\delta_{ik}X^kP_j + i\delta_{jk}X^kP_i \\ J_0 &= \frac{1}{2}(X^iP_i + P_iX^i) \\ J_- &= -\delta_{ij}X^iX^j \\ J_+ &= -\delta^{ij}P_iP_j \\ \Delta &= \frac{1}{2}(xp + px) \end{aligned} \quad (2.19)$$

The generators M_{ij} of $SO(d)$ satisfy the commutation relations

$$[M_{ij}, M_{kl}] = -\delta_{jk}M_{il} + \delta_{ik}M_{jl} + \delta_{jl}M_{ik} - \delta_{il}M_{jk} \quad (2.20)$$

and the generators J_0 and J_\pm of $Sp(2, \mathbb{R})$ satisfy

$$[J_0, J_\pm] = \pm 2iJ_\pm \quad [J_-, J_+] = 4iJ_0. \quad (2.21)$$

Note that the compact generator of this $Sp(2, \mathbb{R})$ is $(J_+ + J_-)$.

The coordinate X^i and momentum P_i operators transform in the vector representation of $SO(d)$ subgroup generated by M_{ij} and form doublets of the symplectic group $Sp(2, \mathbb{R})$:

$$\begin{aligned} [J_0, V^i] &= -iV^i & [J_-, V^i] &= 0 & [J_+, V^i] &= +2i\delta^{ij}U_j \\ [J_0, U_i] &= +iU_i & [J_-, U_i] &= -2i\delta_{ij}V^j & [J_+, U_i] &= 0 \end{aligned} \quad (2.22)$$

There is a normal ordering ambiguity in defining the quantum operator corresponding to the quartic invariant. We shall choose the quantum quartic invariant given in [49]:

$$\begin{aligned}\mathcal{I}_4 = & (\delta_{ij} X^i X^j) (\delta^{ij} P_i P_j) + (\delta^{ij} P_i P_j) (\delta_{ij} X^i X^j) \\ & - (X^i P_i) (P_j X^j) - (P_i X^i) (X^j P_j)\end{aligned}\quad (2.23)$$

In terms of the quartic invariant, the grade +2 generator K_+ of $SO(d+2, 2)$ takes the form

$$K_+ = \frac{1}{2} p^2 + \frac{1}{4x^2} \left(\mathcal{I}_4 + \frac{d^2 + 3}{2} \right). \quad (2.24)$$

Then grade +1 generators are obtained by the commutation of grade -1 generators with K_+ :

$$\tilde{U}_i = -i [U_i, K_+] \quad \tilde{V}^i = -i [V^i, K_+] \quad (2.25)$$

which explicitly read as follows:

$$\begin{aligned}\tilde{U}_i = & p P_i - \frac{1}{2x} \delta_{ij} \delta^{kl} (X^j P_k P_l + P_k P_l X^j) \\ & + \frac{1}{4x} [P_i (X^j P_j + P_j X^j) + (X^j P_j + P_j X^j) P_i] \\ \tilde{V}^i = & p X^i + \frac{1}{2x} \delta^{ij} \delta_{kl} (P_j X^k X^l + X^k X^l P_j) \\ & - \frac{1}{4x} [X^i (X^j P_j + P_j X^j) + (X^j P_j + P_j X^j) X^i]\end{aligned}\quad (2.26)$$

Conversely we also have

$$V^i = i [\tilde{V}^i, K_-] \quad U_i = i [\tilde{U}_i, K_-]. \quad (2.27)$$

The generators in $\mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)}$ subspace form an Heisenberg algebra isomorphic to equation (2.17):

$$[\tilde{V}^i, \tilde{U}_j] = 2i \delta_j^i K_+ \quad (2.28)$$

Commutators $[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(+1)}]$ close into grade zero subspace $\mathfrak{g}^{(0)}$:

$$\begin{aligned}[U_i, \tilde{U}_j] &= -i \delta_{ij} J_+ & [V^i, \tilde{V}^j] &= -i \delta^{ij} J_- \\ [V^i, \tilde{U}_j] &= -2 \delta^{ik} M_{kj} + i \delta_j^i (J_0 + \Delta) \\ [U_i, \tilde{V}^j] &= +2 \delta^{jk} M_{ik} + i \delta_i^j (J_0 - \Delta)\end{aligned}\quad (2.29)$$

Δ is the generator that determines the 5-grading:

$$\begin{aligned}[K_-, K_+] &= i \Delta & [\Delta, K_\pm] &= \pm 2i K_\pm \\ [\Delta, U_i] &= -i U_i & [\Delta, V^i] &= -i V^i \\ [\Delta, \tilde{U}_i] &= +i \tilde{U}_i & [\Delta, \tilde{V}^i] &= +i \tilde{V}^i\end{aligned}\quad (2.30)$$

We note that in this realization, the generators M_{ij} are anti-hermitian and all the other generators of $SO(d+2, 2)$ are hermitian.

The quadratic Casimir operators of subalgebras $\mathfrak{so}(d)$ and $\mathfrak{sp}(2, \mathbb{R})_J$ of grade zero subspace, and $\mathfrak{sp}(2, \mathbb{R})_K$ generated by K_\pm and Δ are given by

$$\begin{aligned} M_{ij}M^{ij} &= -\mathcal{I}_4 - 2d \\ J_-J_+ + J_+J_- - 2(J_0)^2 &= \mathcal{I}_4 + \frac{d^2}{2} \\ K_-K_+ + K_+K_- - \frac{1}{2}\Delta^2 &= \frac{1}{4}\mathcal{I}_4 + \frac{d^2}{8}. \end{aligned} \tag{2.31}$$

They all reduce to the quartic invariant operator \mathcal{I}_4 modulo some additive constants. Furthermore, grade ± 1 generators belonging to the coset

$$\frac{SO(d+2, 2)}{SO(d) \times SO(2, 2)}$$

satisfy the identity

$$U_i \tilde{V}^i + \tilde{V}^i U_i - V^i \tilde{U}_i - \tilde{U}_i V^i = 2\mathcal{I}_4 + d(d+4) \tag{2.32}$$

in the above realization. The above relations prove the existence of a family of degree 2 polynomials in the enveloping algebra of $\mathfrak{so}(d+2, 2)$ that degenerate to a c -number for the minimal unitary realization, in accordance with Joseph's theorem [26]:

$$\begin{aligned} M_{ij}M^{ij} + \kappa_1 \left(J_-J_+ + J_+J_- - 2(J_0)^2 \right) + 4\kappa_2 \left(K_-K_+ + K_+K_- - \frac{1}{2}\Delta^2 \right) \\ - \frac{1}{2}(\kappa_1 + \kappa_2 - 1) \left(U_i \tilde{V}^i + \tilde{V}^i U_i - V^i \tilde{U}_i - \tilde{U}_i V^i \right) \\ = \frac{1}{2}d[d - 4(\kappa_1 + \kappa_2)] \end{aligned} \tag{2.33}$$

The quadratic Casimir of $\mathfrak{so}(d+2, 2)$ corresponds to the choice $2\kappa_1 = 2\kappa_2 = -1$ in (2.33). Hence the eigenvalue of the quadratic Casimir for the minimal unitary representation is equal to $\frac{1}{2}d(d+4)$. This minimal unitary representation is realized over the Hilbert space of square integrable functions in $(d+1)$ variables.

3. Minimal Unitary Realization of $SO(6, 2)$ over the Hilbert Space of L^2 Functions in Five Variables

We shall specialize the minimal unitary realization of $SO(d+2, 2)$ given above to the case of $SO(6, 2)$. The corresponding 5-grading of the Lie algebra of $SO(6, 2)$ is with respect to its subalgebra $\mathfrak{g}^{(0)} = \mathfrak{so}(4) \oplus \mathfrak{sp}(2, \mathbb{R}) \oplus \mathfrak{so}(1, 1)$:

$$\begin{aligned} \mathfrak{so}(6, 2) &= \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus [\mathfrak{so}(4) \oplus \mathfrak{sp}(2, \mathbb{R}) \oplus \mathfrak{so}(1, 1)] \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)} \\ &= K_- \oplus [U_i \oplus V^i] \oplus [M_{ij} \oplus J_{\pm, 0} \oplus \Delta] \oplus [\tilde{U}_i \oplus \tilde{V}^i] \oplus K_+ \end{aligned} \tag{3.1}$$

where $i, j, \dots = 1, 2, 3, 4$.

3.1 The noncompact 3-grading of $SO(6, 2)$ with respect to the subgroup $SO(5, 1) \times SO(1, 1)$

Considered as the six dimensional conformal group, $SO(6, 2)$ has a natural 3-grading with respect to the generator \mathcal{D} of dilatations whose eigenvalues determine the conformal dimensions of operators and states. Let us denote the corresponding 3-graded decomposition of $\mathfrak{so}(6, 2)$ as

$$\mathfrak{so}(6, 2) = \mathfrak{N}^- \oplus \mathfrak{N}^0 \oplus \mathfrak{N}^+ \quad (3.2)$$

where $\mathfrak{N}^0 = \mathfrak{so}(5, 1) \oplus \mathfrak{so}(1, 1)_{\mathcal{D}}$ with the subalgebra $\mathfrak{so}(5, 1)$ in \mathfrak{N}^0 representing the Lorentz algebra in six dimensions. The *noncompact* dilatation generator $\mathfrak{so}(1, 1)_{\mathcal{D}}$ is given by

$$\mathcal{D} = \frac{1}{2}(\Delta + J_0) = \frac{1}{4}(xp + px + X^i P_i + P_i X^i) \quad (3.3)$$

and the generators belonging to \mathfrak{N}^\pm and \mathfrak{N}^0 are as follows:

$$\begin{aligned} \mathfrak{N}^- &= K_- \oplus J_- \oplus V^i \\ \mathfrak{N}^0 &= \mathcal{D} \oplus \frac{1}{2}(\Delta - J_0) \oplus M_{ij} \oplus U_i \oplus \tilde{V}^i \\ \mathfrak{N}^+ &= K_+ \oplus J_+ \oplus \tilde{U}_i \end{aligned} \quad (3.4)$$

The Lorentz generators $\mathcal{M}_{\mu\nu}$ ($\mu, \nu, \dots = 0, 1, 2, \dots, 5$) are given by

$$\begin{aligned} \mathcal{M}_{0i} &= \frac{1}{2}(U_i + \delta_{ij}\tilde{V}^j) & \mathcal{M}_{05} &= \frac{1}{2}(\Delta - J_0) \\ \mathcal{M}_{ij} &= -i M_{ij} & \mathcal{M}_{i5} &= \frac{1}{2}(U_i - \delta_{ij}\tilde{V}^j) \end{aligned} \quad (3.5)$$

and satisfy the $\mathfrak{so}(5, 1)$ commutation relations

$$[\mathcal{M}_{\mu\nu}, \mathcal{M}_{\rho\tau}] = i(\eta_{\nu\rho}\mathcal{M}_{\mu\tau} - \eta_{\mu\rho}\mathcal{M}_{\nu\tau} - \eta_{\nu\tau}\mathcal{M}_{\mu\rho} + \eta_{\mu\tau}\mathcal{M}_{\nu\rho}) \quad (3.6)$$

where $\eta_{\mu\nu} = \text{diag}(-, +, +, +, +, +)$. The six translation generators \mathcal{P}_μ ($\mu = 0, 1, 2, \dots, 5$) of the conformal group $SO(6, 2)$ are given by

$$\mathcal{P}_0 = K_+ - \frac{1}{2}J_+ \quad \mathcal{P}_i = \tilde{U}_i \quad (i = 1, 2, 3, 4) \quad \mathcal{P}_5 = K_+ + \frac{1}{2}J_+ \quad (3.7)$$

and the special conformal generators \mathcal{K}_μ ($\mu = 0, 1, 2, \dots, 5$) are given by

$$\mathcal{K}_0 = -\frac{1}{2}J_- + K_- \quad \mathcal{K}_i = -V^i \quad (i = 1, 2, 3, 4) \quad \mathcal{K}_5 = -\frac{1}{2}J_- - K_- \quad (3.8)$$

These generators satisfy the commutation relations of $SO(6, 2)$ as the six dimensional conformal algebra:

$$\begin{aligned} [\mathcal{M}_{\mu\nu}, \mathcal{M}_{\rho\tau}] &= i(\eta_{\nu\rho}\mathcal{M}_{\mu\tau} - \eta_{\mu\rho}\mathcal{M}_{\nu\tau} - \eta_{\nu\tau}\mathcal{M}_{\mu\rho} + \eta_{\mu\tau}\mathcal{M}_{\nu\rho}) \\ [\mathcal{P}_\mu, \mathcal{M}_{\nu\rho}] &= i(\eta_{\mu\nu}\mathcal{P}_\rho - \eta_{\mu\rho}\mathcal{P}_\nu) \\ [\mathcal{K}_\mu, \mathcal{M}_{\nu\rho}] &= i(\eta_{\mu\nu}\mathcal{K}_\rho - \eta_{\mu\rho}\mathcal{K}_\nu) \\ [\mathcal{D}, \mathcal{M}_{\mu\nu}] &= [\mathcal{P}_\mu, \mathcal{P}_\nu] = [\mathcal{K}_\mu, \mathcal{K}_\nu] = 0 \\ [\mathcal{D}, \mathcal{P}_\mu] &= +i\mathcal{P}_\mu \quad [\mathcal{D}, \mathcal{K}_\mu] = -i\mathcal{K}_\mu \\ [\mathcal{P}_\mu, \mathcal{K}_\nu] &= 2i(\eta_{\mu\nu}\mathcal{D} + \mathcal{M}_{\mu\nu}) \end{aligned} \quad (3.9)$$

We should note that the $6D$ Poincaré mass operator vanishes identically

$$\mathcal{M}^2 = \eta_{\mu\nu} \mathcal{P}^\mu \mathcal{P}^\nu = 0 \quad (3.10)$$

for the minimal unitary realization given above. Hence the minimal unitary representation of $SO(6, 2)$ corresponds to a massless representation as a six dimensional conformal group. We shall refer to the above 3-graded decomposition as the *noncompact* 3-grading.

3.2 The compact 3-grading of $SO(6, 2)$ with respect to the subgroup $SO(6) \times SO(2)$

The Lie algebra $\mathfrak{so}(6, 2)$ has a 3-grading with respect to its maximal compact subalgebra $\mathfrak{C}^0 = \mathfrak{so}(6) \oplus \mathfrak{so}(2)$, determined by the $\mathfrak{so}(2)$ generator

$$H = \frac{1}{2} \left[(K_+ + K_-) - \frac{1}{2} (J_+ + J_-) \right] \quad (3.11)$$

such that

$$\mathfrak{so}(6, 2) = \mathfrak{C}^- \oplus [\mathfrak{so}(6) \oplus \mathfrak{so}(2)] \oplus \mathfrak{C}^+ \quad (3.12)$$

and satisfy

$$[H, \mathfrak{C}^+] = +\mathfrak{C}^+ \quad [H, \mathfrak{C}^-] = -\mathfrak{C}^- . \quad (3.13)$$

In this decomposition:

$$\begin{aligned} \mathfrak{C}^0 = \mathfrak{so}(6) \oplus \mathfrak{so}(2) &= \left[M_{ij} \oplus \left((K_+ + K_-) + \frac{1}{2} (J_+ + J_-) \right) \oplus \left(U_i - \delta_{ij} \tilde{V}^j \right) \right. \\ &\quad \left. \left(\tilde{U}_i + \delta_{ij} V^j \right) \right] \oplus \frac{1}{2} \left[(K_+ + K_-) - \frac{1}{2} (J_+ + J_-) \right] \\ \mathfrak{C}^+ &= [\Delta - i(K_+ - K_-)] \oplus \left[J_0 + \frac{i}{2} (J_+ - J_-) \right] \oplus \left[\frac{1}{2} (U_i + \delta_{ij} \tilde{V}^j) - \frac{i}{2} (\tilde{U}_i - \delta_{ij} V^j) \right] \\ \mathfrak{C}^- &= [\Delta + i(K_+ - K_-)] \oplus \left[J_0 - \frac{i}{2} (J_+ - J_-) \right] \oplus \left[\frac{1}{2} (U_i + \delta_{ij} \tilde{V}^j) + \frac{i}{2} (\tilde{U}_i - \delta_{ij} V^j) \right] \end{aligned} \quad (3.14)$$

Note that in the above 3-grading, the operators belonging to \mathfrak{C}^+ are Hermitian conjugates of those belonging to \mathfrak{C}^- . In the corresponding minimal unitary realization one takes only the hermitian linear combinations of these operators as generators of $\mathfrak{so}(6, 2)$. The generator H is the conformal Hamiltonian or the AdS energy depending on whether one is considering $SO(6, 2)$ as six dimensional conformal group or the seven dimensional AdS group. We shall refer to this grading as the *compact* 3-grading.

The $\mathfrak{so}(6)$ generators \tilde{M}_{MN} ($M, N, \dots = 1, 2, \dots, 6$) in grade zero subspace \mathfrak{C}^0 are given by

$$\begin{aligned} \tilde{M}_{ij} &= i M_{ij} & \tilde{M}_{i5} &= \frac{1}{2} (U_i - \delta_{ij} \tilde{V}^j) \\ \tilde{M}_{i6} &= \frac{1}{2} (\tilde{U}_i + \delta_{ij} V^j) & \tilde{M}_{56} &= \frac{1}{2} \left[(K_+ + K_-) + \frac{1}{2} (J_+ + J_-) \right] \end{aligned} \quad (3.15)$$

and satisfy the $\mathfrak{so}(6)$ algebra

$$\left[\widetilde{M}_{MN}, \widetilde{M}_{PQ} \right] = i \left(\delta_{NP} \widetilde{M}_{MQ} - \delta_{MP} \widetilde{M}_{NQ} - \delta_{NQ} \widetilde{M}_{MP} + \delta_{MQ} \widetilde{M}_{NP} \right). \quad (3.16)$$

To give the decomposition of the minrep of $SO(6, 2)$ into its K-finite vectors of its maximal compact subgroup $SO(6) \times SO(2)$ we shall define the oscillators

$$c_i = \frac{1}{\sqrt{2}} (X^i + i P_i) \quad c_i^\dagger = \frac{1}{\sqrt{2}} (X^i - i P_i) \quad (3.17)$$

or conversely

$$X^i = \frac{1}{\sqrt{2}} (c_i^\dagger + c_i) \quad P_i = \frac{i}{\sqrt{2}} (c_i^\dagger - c_i). \quad (3.18)$$

These oscillators satisfy the commutation relations

$$\left[c_i, c_j^\dagger \right] = \delta_{ij}. \quad (3.19)$$

The quartic invariant operator \mathcal{I}_4 takes on a simple form in terms of these oscillators:

$$\begin{aligned} \mathcal{I}_4 &= - \left(c_i^\dagger c_j - c_j^\dagger c_i \right)^2 - 8 \\ &= -M_{ij} M_{ij} - 8 \end{aligned} \quad (3.20)$$

The $\mathfrak{so}(2)$ generator in \mathfrak{C}^0 , that determines the 3-grading and plays the role of the AdS energy [22, 59, 60], is given in terms of x, p and oscillators c_i, c_i^\dagger as:

$$\begin{aligned} H &= \frac{1}{2} \left[(K_+ + K_-) - \frac{1}{2} (J_+ + J_-) \right] \\ &= \frac{1}{4} (x^2 + p^2) + \frac{1}{2} c_i^\dagger c_i - \frac{1}{8x^2} \left(c_i^\dagger c_j - c_j^\dagger c_i \right)^2 + \frac{3}{16x^2} + 1 \end{aligned} \quad (3.21)$$

We can also write the $\mathfrak{so}(6)$ generators \widetilde{M}_{MN} in terms of these oscillators as follows:

$$\begin{aligned} \widetilde{M}_{ij} &= i M_{ij} \\ &= i \left(c_i^\dagger c_j - c_j^\dagger c_i \right) \\ \widetilde{M}_{i5} &= \frac{1}{2} \left(U_i - \delta_{ij} \widetilde{V}^j \right) \\ &= \frac{i}{2\sqrt{2}} (x + i p) c_i^\dagger - \frac{i}{2\sqrt{2}} (x - i p) c_i - \frac{i}{2\sqrt{2}x} \left(c_i^\dagger c_j - c_j^\dagger c_i \right) \left(c_j^\dagger + c_j \right) + \frac{3i}{4\sqrt{2}x} \left(c_i^\dagger + c_i \right) \\ \widetilde{M}_{i6} &= \frac{1}{2} \left(\widetilde{U}_i + \delta_{ij} V^j \right) \\ &= \frac{1}{2\sqrt{2}} (x + i p) c_i^\dagger + \frac{1}{2\sqrt{2}} (x - i p) c_i - \frac{1}{2\sqrt{2}x} \left(c_i^\dagger c_j - c_j^\dagger c_i \right) \left(c_j^\dagger - c_j \right) + \frac{3}{4\sqrt{2}x} \left(c_i^\dagger - c_i \right) \\ \widetilde{M}_{56} &= \frac{1}{2} \left[(K_+ + K_-) + \frac{1}{2} (J_+ + J_-) \right] \\ &= \frac{1}{4} (x^2 + p^2) - \frac{1}{2} c_i^\dagger c_i - \frac{1}{8x^2} \left(c_i^\dagger c_j - c_j^\dagger c_i \right)^2 + \frac{3}{16x^2} - 1 \end{aligned} \quad (3.22)$$

Six operators that belong to the grade +1 subspace \mathfrak{E}^+ have the following form in terms of these oscillators:

$$\begin{aligned}
\frac{1}{2} \left[\left(U_i + \delta_{ij} \tilde{V}^j \right) - i \left(\tilde{U}_i - \delta_{ij} V^j \right) \right] &= \frac{i}{\sqrt{2}} (x - ip) c_i^\dagger \\
&\quad + \frac{i}{2\sqrt{2}x} \left[c_j^\dagger (c_i^\dagger c_j - c_j^\dagger c_i) + (c_i^\dagger c_j - c_j^\dagger c_i) c_j^\dagger \right] \\
J_0 + \frac{i}{2} (J_+ - J_-) &= i c_i^\dagger c_i^\dagger \\
\Delta - i (K_+ - K_-) &= \frac{i}{2} (x - ip)^2 + \frac{i}{4x^2} \left[(c_i^\dagger c_j - c_j^\dagger c_i)^2 + \frac{9}{2} \right]
\end{aligned} \tag{3.23}$$

and those that belong to the grade -1 subspace \mathfrak{E}^- are given by

$$\begin{aligned}
\frac{1}{2} \left[\left(U_i + \delta_{ij} \tilde{V}^j \right) + i \left(\tilde{U}_i - \delta_{ij} V^j \right) \right] &= -\frac{i}{\sqrt{2}} (x + ip) c_i \\
&\quad + \frac{i}{2\sqrt{2}x} \left[c_j (c_i^\dagger c_j - c_j^\dagger c_i) + (c_i^\dagger c_j - c_j^\dagger c_i) c_j \right] \\
J_0 - \frac{i}{2} (J_+ - J_-) &= -i c_i c_i \\
\Delta + i (K_+ - K_-) &= -\frac{i}{\sqrt{2}} (x + ip)^2 - \frac{i}{4x^2} \left[(c_i^\dagger c_j - c_j^\dagger c_i)^2 + \frac{9}{2} \right].
\end{aligned} \tag{3.24}$$

One could also give a decomposition of $SO(6, 2)$ with respect to the subgroup $SO(4) \times SO(2, 2)$, which we present in appendix A.

4. Minimal Unitary Representation of $SO^*(8)$

The groups $SO(d+2, 2)$ have supersymmetric extensions which are in general supergroups of the form $OSp(d+2, 2 | 2n, \mathbb{R})$ with even subgroups $SO(d+2, 2) \times Sp(2n, \mathbb{R})$. The supergroups whose even subgroups are products of two simple noncompact groups do not, in general, admit any unitary representations. Furthermore, if the group $SO(d+2, 2)$ is considered as a conformal group in $(d+2)$ dimensions or as anti-de Sitter group in $(d+3)$ dimensions, its factor group in its supersymmetric extension is the R -symmetry group which must be compact [64]. Remarkably, either the existence of exceptional superalgebras or certain special isomorphisms allow such possibilities for special values of d . The group $SO(5, 2)$ has an extension to the exceptional supergroup $F(4)$ with even subgroup $SO(5, 2) \times SU(2)$ which admits positive energy unitary representations. The covering group of $SO(4, 2)$ is the group $SU(2, 2)$ which extends to an infinite family of supergroups $SU(2, 2 | N)$ with even subgroups $SU(2, 2) \times U(N)$ that admit positive energy unitary representations. Similarly isomorphism of $SO(3, 2)$ to $Sp(4, \mathbb{R})$ allows extension to supergroups $OSp(N | 4, \mathbb{R})$ with even subgroups $Sp(4, \mathbb{R}) \times SO(N)$ that admit positive energy unitary representations. Since $SO(2, 2)$ is not simple, one finds a rich family of supersymmetric extensions that admit positive energy unitary representations that were studied in [65]. Similarly, the Lie algebra of $SO(6, 2)$

is isomorphic to that of $SO^*(8)$ which have extensions to supergroups $OSp(8^*|2N)$ with even subgroups $SO^*(8) \times USp(2N)$ that admit positive energy unitary representations.² Hence we will now study the minimal unitary realizations of $SO^*(8)$ and their supersymmetric extensions in the subsequent sections.

4.1 The 5-grading of $SO^*(8)$ with respect to the subgroup $SO^*(4) \times SU(2) \times SO(1, 1)$

The noncompact Lie algebra $\mathfrak{so}^*(8)$ has a 5-grading with respect to its subalgebra $\mathfrak{so}^*(4) \oplus \mathfrak{su}(2) \oplus \mathfrak{so}(1, 1)$, where the $\mathfrak{so}(1, 1)$ generator Δ defines the 5-grading [52]:

$$\mathfrak{so}^*(8) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus [\mathfrak{so}^*(4) \oplus \mathfrak{su}(2) \oplus \Delta] \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)} \quad (4.1)$$

such that

$$[\Delta, \mathfrak{g}^{(m)}] = m \mathfrak{g}^{(m)} \quad (4.2)$$

In this decomposition, $\mathfrak{g}^{(\pm 2)}$ subspaces are one-dimensional, and $\mathfrak{g}^{(\pm 1)}$ subspaces transform in the $(\mathbf{4}, \mathbf{2})$ dimensional representation of $SO^*(4) \times SU(2)$. Since $SO^*(4) = SU(1, 1) \times SU(2)$, the grade zero subalgebra $SU(1, 1) \times SU(2) \times SU(2) \times SO(1, 1)$ is also isomorphic to that of $SO(6, 2)$.

For the study of the minrep of $SO^*(8)$ we shall relabel the oscillators introduced in the previous sections as a_m, b_m and their hermitian conjugates $a^m = (a_m)^\dagger, b^m = (b_m)^\dagger$ ($m, n, \dots = 1, 2$)

$$\begin{aligned} a_m &= \frac{1}{\sqrt{2}} (X^m + i P_m) & a^m &= \frac{1}{\sqrt{2}} (X^m - i P_m) \\ b_m &= \frac{1}{\sqrt{2}} (X^{2+m} + i P_{2+m}) & b^m &= \frac{1}{\sqrt{2}} (X^{2+m} - i P_{2+m}) \end{aligned} \quad (4.3)$$

so that

$$\begin{aligned} X^m &= \frac{1}{\sqrt{2}} (a^m + a_m) & P_m &= \frac{i}{\sqrt{2}} (a^m - a_m) \\ X^{2+m} &= \frac{1}{\sqrt{2}} (b^m + b_m) & P_{2+m} &= \frac{i}{\sqrt{2}} (b^m - b_m) . \end{aligned} \quad (4.4)$$

They satisfy the commutation relations

$$[a_m, a^n] = \delta_m^n \quad [b_m, b^n] = \delta_m^n . \quad (4.5)$$

Then the generators of $\mathfrak{su}(2)$ of $\mathfrak{g}^{(0)}$ that commute with $\mathfrak{so}^*(4)$ can be realized as follows:

$$S_+ = a^m b_m \quad S_- = (S_+)^\dagger = a_m b^m \quad S_0 = \frac{1}{2} (N_a - N_b) \quad (4.6)$$

where $N_a = a^m a_m$ and $N_b = b^m b_m$ are the respective number operators. We denote this subalgebra as $\mathfrak{su}(2)_S$. Its generators satisfy:

$$[S_+, S_-] = 2 S_0 \quad [S_0, S_\pm] = \pm S_\pm \quad (4.7)$$

²As such it belongs to an infinite family of supergroups $OSp(2M^*|2N)$ with even subgroups $SO^*(2M) \times USp(2N)$ whose positive energy unitary representations were studied in [63].

The quadratic Casimir of $\mathfrak{su}(2)_S$ is given by

$$\begin{aligned} \mathcal{C}_2 [\mathfrak{su}(2)_S] &= \mathcal{S}^2 = S_0^2 + \frac{1}{2} (S_+ S_- + S_- S_+) \\ &= \frac{1}{2} (N_a + N_b) \left[\frac{1}{2} (N_a + N_b) + 1 \right] - 2a^{[m} b^{n]} a_{[m} b_{n]} \end{aligned} \quad (4.8)$$

where square bracketing $a_{[m} b_{n]} = \frac{1}{2} (a_m b_n - a_n b_m)$ represents antisymmetrization of weight one.

We shall label the simple factors of $\mathfrak{so}^*(4)$ subalgebra that commutes with $\mathfrak{su}(2)_S$ as

$$\mathfrak{so}^*(4) = \mathfrak{su}(2)_A \oplus \mathfrak{su}(1, 1)_N \quad (4.9)$$

and denote the generators of $\mathfrak{su}(2)_A$ and $\mathfrak{su}(1, 1)_N$ as $A_{\pm, 0}$ and $N_{\pm, 0}$, respectively. In terms of the above a -type and b -type oscillators, these generators have the following realization:

$$\begin{aligned} A_+ &= a^1 a_2 + b^1 b_2 & N_+ &= a^1 b^2 - a^2 b^1 \\ A_- &= (A_+)^{\dagger} = a_1 a^2 + b_1 b^2 & N_- &= (N_+)^{\dagger} = a_1 b_2 - a_2 b_1 \\ A_0 &= \frac{1}{2} (a^1 a_1 - a^2 a_2 + b^1 b_1 - b^2 b_2) & N_0 &= \frac{1}{2} (N_a + N_b) + 1 \end{aligned} \quad (4.10)$$

They satisfy the commutation relations:

$$\begin{aligned} [A_+, A_-] &= 2 A_0 & [N_-, N_+] &= 2 N_0 \\ [A_0, A_{\pm}] &= \pm A_{\pm} & [N_0, N_{\pm}] &= \pm N_{\pm} \end{aligned} \quad (4.11)$$

The quadratic Casimirs of these subalgebras

$$\begin{aligned} \mathcal{C}_2 [\mathfrak{su}(2)_A] &= \mathcal{A}^2 = A_0^2 + \frac{1}{2} (A_+ A_- + A_- A_+) \\ \mathcal{C}_2 [\mathfrak{su}(1, 1)_N] &= \mathcal{N}^2 = N_0^2 - \frac{1}{2} (N_+ N_- + N_- N_+) \end{aligned} \quad (4.12)$$

coincide and are equal to that of $\mathfrak{su}(2)_S$:

$$\mathcal{S}^2 = \mathcal{A}^2 = \mathcal{N}^2 \quad (4.13)$$

The transformations relating the $SO^*(8)$ oscillators a_m and b_m to the $SO(6, 2)$ oscillators c_i are given in Appendix C.

The generator that defines the 5-grading can be written as

$$\Delta = \frac{1}{2} (xp + px) \quad (4.14)$$

and the $\mathfrak{g}^{(\pm 2)}$ generators are realized as

$$K_- = \frac{1}{2} x^2 \quad K_+ = \frac{1}{2} p^2 + \frac{1}{4 x^2} \left(8 \mathcal{S}^2 + \frac{3}{2} \right). \quad (4.15)$$

These three generators form another $\mathfrak{su}(1,1)_K$ subalgebra

$$[K_-, K_+] = i \Delta \quad [\Delta, K_\pm] = \pm 2i K_\pm \quad (4.16)$$

with the quadratic Casimir operator

$$\mathcal{C}_2[\mathfrak{su}(1,1)_K] = \mathcal{K}^2 = \frac{1}{2}(K_+ K_- + K_- K_+) - \frac{1}{4} \Delta^2 = \mathcal{S}^2. \quad (4.17)$$

The eight generators that are in $\mathfrak{g}^{(-1)}$ subspace take the form

$$\begin{aligned} U_m &= x a_m & U^m &= x a^m \\ V_m &= x b_m & V^m &= x b^m \end{aligned} \quad (4.18)$$

and together with K_- form an Heisenberg algebra:

$$\begin{aligned} [U_m, U^n] &= [V_m, V^n] = 2 \delta_m^n K_- \\ [U_m, U_n] &= [V_m, V_n] = 0 \end{aligned} \quad (4.19)$$

The generators in $\mathfrak{g}^{(+1)}$ are obtained from the commutators $[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(+2)}]$:

$$\begin{aligned} \tilde{U}_m &= i [U_m, K_+] & \tilde{U}^m &= (\tilde{U}_m)^\dagger = i [U^m, K_+] \\ \tilde{V}_m &= i [V_m, K_+] & \tilde{V}^m &= (\tilde{V}_m)^\dagger = i [V^m, K_+] \end{aligned} \quad (4.20)$$

Explicitly they are given by

$$\begin{aligned} \tilde{U}_m &= -p a_m + \frac{2i}{x} \left[\left(S_0 + \frac{3}{4} \right) a_m + S_- b_m \right] \\ \tilde{U}^m &= -p a^m - \frac{2i}{x} \left[\left(S_0 - \frac{3}{4} \right) a^m + S_+ b^m \right] \\ \tilde{V}_m &= -p b_m - \frac{2i}{x} \left[\left(S_0 - \frac{3}{4} \right) b_m - S_+ a_m \right] \\ \tilde{V}^m &= -p b^m + \frac{2i}{x} \left[\left(S_0 + \frac{3}{4} \right) b^m - S_- a^m \right] \end{aligned} \quad (4.21)$$

and also form another Heisenberg algebra with K_+ as its “central charge”:

$$\begin{aligned} [\tilde{U}_m, \tilde{U}^n] &= [\tilde{V}_m, \tilde{V}^n] = 2 \delta_m^n K_+ \\ [\tilde{U}_m, \tilde{U}_n] &= [\tilde{V}_m, \tilde{V}_n] = 0 \end{aligned} \quad (4.22)$$

The commutators $[\mathfrak{g}^{(-2)}, \mathfrak{g}^{(+1)}]$ take the following form:

$$\begin{aligned} [\tilde{U}_m, K_-] &= i U_m & [\tilde{U}^m, K_-] &= i U^m \\ [\tilde{V}_m, K_-] &= i V_m & [\tilde{V}^m, K_-] &= i V^m \end{aligned} \quad (4.23)$$

Finally, the non-vanishing commutators of the form $[\mathfrak{g}^{(-1)}, \mathfrak{g}^{(+1)}]$ are as follows:

$$\begin{aligned}
[U_m, \tilde{U}^n] &= -\delta_m^n \Delta - 2i \delta_m^n N_0 - 2i \delta_m^n S_0 - 2i A_m^n \\
[V_m, \tilde{V}^n] &= -\delta_m^n \Delta - 2i \delta_m^n N_0 + 2i \delta_m^n S_0 - 2i A_m^n \\
[U_m, \tilde{V}^n] &= -2i \delta_m^n S_- & [V_m, \tilde{U}^n] &= -2i \delta_m^n S_+ \\
[U_m, \tilde{V}_n] &= -2i \epsilon_{mn} N_- & [V_m, \tilde{U}_n] &= +2i \epsilon_{mn} N_-
\end{aligned} \tag{4.24}$$

where we have labeled the generators of $\mathfrak{su}(2)_A$ as A_n^m :

$$A_1^1 = -A_2^2 = A_0 \quad A_1^2 = A_+ \quad A_2^1 = (A_1^2)^\dagger = A_- \tag{4.25}$$

and denoted the completely antisymmetric tensor by ϵ_{mn} ($\epsilon_{12} = +1$).

With the generators defined above, the 5-grading of the Lie algebra $\mathfrak{so}^*(8)$, defined by Δ , takes the form:

$$\begin{aligned}
\mathfrak{so}^*(8) &= \mathbf{1} \oplus (\mathbf{4}, \mathbf{2}) \oplus [\mathfrak{su}(2)_A \oplus \mathfrak{su}(1, 1)_N \oplus \mathfrak{su}(2)_S \oplus \mathfrak{so}(1, 1)_\Delta] \oplus (\mathbf{4}, \mathbf{2}) \oplus \mathbf{1} \\
&= K_- \oplus [U_m, U^m, V_m, V^m] \oplus [A_{\pm, 0} \oplus N_{\pm, 0} \oplus S_{\pm, 0} \oplus \Delta] \\
&\quad \oplus [\tilde{U}_m, \tilde{U}^m, \tilde{V}_m, \tilde{V}^m] \oplus K_+
\end{aligned} \tag{4.26}$$

As expected, the quadratic Casimir of $\mathfrak{so}^*(8)$, given by

$$\mathcal{C}_2[\mathfrak{so}^*(8)] = \mathcal{C}_2[\mathfrak{su}(2)_S] + \mathcal{C}_2[\mathfrak{su}(2)_A] + \mathcal{C}_2[\mathfrak{su}(1, 1)_N] + \mathcal{C}_2[\mathfrak{su}(1, 1)_K] - \frac{i}{4} \mathcal{F}(U, V) \tag{4.27}$$

where

$$\begin{aligned}
\mathcal{F}(U, V) &= (U_m \tilde{U}^m + V_m \tilde{V}^m + \tilde{U}^m U_m + \tilde{V}^m V_m) \\
&\quad - (U^m \tilde{U}_m + V^m \tilde{V}_m + \tilde{U}_m U^m + \tilde{V}_m V^m)
\end{aligned} \tag{4.28}$$

reduces to a c -number, -4 .

4.2 The noncompact 3-grading of $SO^*(8)$ with respect to the subgroup $SU^*(4) \times SO(1, 1)$

Considered as the six dimensional conformal group, $SO^*(8)$ has a noncompact 3-grading determined by the dilatation generator \mathcal{D} :

$$\mathfrak{so}^*(8) = \mathfrak{N}^- \oplus \mathfrak{N}^0 \oplus \mathfrak{N}^+ \tag{4.29}$$

where $\mathfrak{N}^0 = \mathfrak{su}^*(4) \oplus \mathfrak{so}(1, 1)_{\mathcal{D}}$ and

$$\mathcal{D} = \frac{1}{2} [\Delta - i(N_+ - N_-)] \tag{4.30}$$

The generators that belong to \mathfrak{N}^\pm and \mathfrak{N}^0 subspaces are as follows:

$$\begin{aligned}
\mathfrak{N}^- &= K_- \oplus \left[N_0 - \frac{1}{2}(N_+ + N_-) \right] \\
&\quad \oplus (U^1 - V_2) \oplus (U^2 + V_1) \oplus (V^1 + U_2) \oplus (V^2 - U_1) \\
\mathfrak{N}^0 &= \mathcal{D} \oplus \frac{1}{2}[\Delta + i(N_+ - N_-)] \oplus S_{\pm,0} \oplus A_{\pm,0} \\
&\quad \oplus (U^1 + V_2) \oplus (U^2 - V_1) \oplus (V^1 - U_2) \oplus (V^2 + U_1) \\
&\quad \oplus (\tilde{U}^1 - \tilde{V}_2) \oplus (\tilde{U}^2 + \tilde{V}_1) \oplus (\tilde{V}^1 + \tilde{U}_2) \oplus (\tilde{V}^2 - \tilde{U}_1) \\
\mathfrak{N}^+ &= K_+ \oplus \left[N_0 + \frac{1}{2}(N_+ + N_-) \right] \\
&\quad \oplus (\tilde{U}^1 + \tilde{V}_2) \oplus (\tilde{U}^2 - \tilde{V}_1) \oplus (\tilde{V}^1 - \tilde{U}_2) \oplus (\tilde{V}^2 + \tilde{U}_1)
\end{aligned} \tag{4.31}$$

Since $\mathfrak{su}^*(4) \simeq \mathfrak{so}(5, 1)$, we find that the Lorentz generators $\mathcal{M}_{\mu\nu}$ ($\mu, \nu, \dots = 0, 1, 2, \dots, 5$) in six dimensions are given by:

$$\begin{aligned}
\mathcal{M}_{01} &= \frac{1}{4} \left[(U^1 + V_2) + (V^2 + U_1) + i(\tilde{U}^1 - \tilde{V}_2) + i(\tilde{V}^2 - \tilde{U}_1) \right] \\
\mathcal{M}_{02} &= \frac{i}{4} \left[(U^1 + V_2) - (V^2 + U_1) + i(\tilde{U}^1 - \tilde{V}_2) - i(\tilde{V}^2 - \tilde{U}_1) \right] \\
\mathcal{M}_{03} &= \frac{i}{4} \left[(U^2 - V_1) + (V^1 - U_2) + i(\tilde{U}^2 + \tilde{V}_1) + i(\tilde{V}^1 + \tilde{U}_2) \right] \\
\mathcal{M}_{04} &= -\frac{1}{4} \left[(U^2 - V_1) - (V^1 - U_2) + i(\tilde{U}^2 + \tilde{V}_1) - i(\tilde{V}^1 + \tilde{U}_2) \right]
\end{aligned} \tag{4.32a}$$

$$\begin{aligned}
\mathcal{M}_{15} &= \frac{1}{4} \left[(U^1 + V_2) + (V^2 + U_1) - i(\tilde{U}^1 - \tilde{V}_2) - i(\tilde{V}^2 - \tilde{U}_1) \right] \\
\mathcal{M}_{25} &= \frac{i}{4} \left[(U^1 + V_2) - (V^2 + U_1) - i(\tilde{U}^1 - \tilde{V}_2) + i(\tilde{V}^2 - \tilde{U}_1) \right] \\
\mathcal{M}_{35} &= \frac{i}{4} \left[(U^2 - V_1) + (V^1 - U_2) - i(\tilde{U}^2 + \tilde{V}_1) - i(\tilde{V}^1 + \tilde{U}_2) \right] \\
\mathcal{M}_{45} &= -\frac{1}{4} \left[(U^2 - V_1) - (V^1 - U_2) - i(\tilde{U}^2 + \tilde{V}_1) + i(\tilde{V}^1 + \tilde{U}_2) \right]
\end{aligned} \tag{4.32b}$$

$$\begin{aligned}
\mathcal{M}_{12} &= S_0 + A_0 & \mathcal{M}_{13} &= \frac{1}{2}(S_+ + S_- + A_+ + A_-) \\
\mathcal{M}_{14} &= \frac{i}{2}(S_+ - S_- - A_+ + A_-) & \mathcal{M}_{23} &= \frac{i}{2}(S_+ - S_- + A_+ - A_-) \\
\mathcal{M}_{24} &= -\frac{1}{2}(S_+ + S_- - A_+ - A_-) & \mathcal{M}_{34} &= S_0 - A_0
\end{aligned} \tag{4.32c}$$

$$\mathcal{M}_{05} = \frac{1}{2}[\Delta + i(N_+ - N_-)] \tag{4.32d}$$

These Lorentz generators satisfy the $\mathfrak{so}(5, 1)$ commutation relations

$$[\mathcal{M}_{\mu\nu}, \mathcal{M}_{\rho\tau}] = i(\eta_{\nu\rho}\mathcal{M}_{\mu\tau} - \eta_{\mu\rho}\mathcal{M}_{\nu\tau} - \eta_{\nu\tau}\mathcal{M}_{\mu\rho} + \eta_{\mu\tau}\mathcal{M}_{\nu\rho}) \tag{4.33}$$

where $\eta_{\mu\nu} = \text{diag}(-, +, +, +, +, +)$. The six generators of grade +1 space are the momentum generators \mathcal{P}_μ , and the six generators of grade -1 space are the special conformal transformations \mathcal{K}_μ ($\mu = 0, 1, 2, \dots, 5$):

$$\begin{aligned}
\mathcal{P}_0 &= K_+ + \left[N_0 + \frac{1}{2} (N_+ + N_-) \right] & \mathcal{K}_0 &= \left[N_0 - \frac{1}{2} (N_+ + N_-) \right] + K_- \\
\mathcal{P}_1 &= -\frac{1}{2} \left[(\tilde{U}^1 + \tilde{V}_2) + (\tilde{V}^2 + \tilde{U}_1) \right] & \mathcal{K}_1 &= \frac{i}{2} [(U^1 - V_2) + (V^2 - U_1)] \\
\mathcal{P}_2 &= -\frac{i}{2} \left[(\tilde{U}^1 + \tilde{V}_2) - (\tilde{V}^2 + \tilde{U}_1) \right] & \mathcal{K}_2 &= -\frac{1}{2} [(U^1 - V_2) - (V^2 - U_1)] \\
\mathcal{P}_3 &= -\frac{i}{2} \left[(\tilde{U}^2 - \tilde{V}_1) + (\tilde{V}^1 - \tilde{U}_2) \right] & \mathcal{K}_3 &= -\frac{1}{2} [(U^2 + V_1) + (V^1 + U_2)] \\
\mathcal{P}_4 &= \frac{1}{2} \left[(\tilde{U}^2 - \tilde{V}_1) - (\tilde{V}^1 - \tilde{U}_2) \right] & \mathcal{K}_4 &= -\frac{i}{2} [(U^2 + V_1) - (V^1 + U_2)] \\
\mathcal{P}_5 &= K_+ - \left[N_0 + \frac{1}{2} (N_+ + N_-) \right] & \mathcal{K}_5 &= \left[N_0 - \frac{1}{2} (N_+ + N_-) \right] - K_-
\end{aligned} \tag{4.34}$$

Together with the generators $\mathcal{M}_{\mu\nu}$ and \mathcal{D} , they satisfy the commutation relations:

$$\begin{aligned}
[\mathcal{D}, \mathcal{P}_\mu] &= +i \mathcal{P}_\mu & [\mathcal{D}, \mathcal{K}_\mu] &= -i \mathcal{K}_\mu \\
[\mathcal{D}, \mathcal{M}_{\mu\nu}] &= [\mathcal{P}_\mu, \mathcal{P}_\nu] = [\mathcal{K}_\mu, \mathcal{K}_\nu] = 0 \\
[\mathcal{P}_\mu, \mathcal{M}_{\nu\rho}] &= i (\eta_{\mu\nu} \mathcal{P}_\rho - \eta_{\mu\rho} \mathcal{P}_\nu) \\
[\mathcal{K}_\mu, \mathcal{M}_{\nu\rho}] &= i (\eta_{\mu\nu} \mathcal{K}_\rho - \eta_{\mu\rho} \mathcal{K}_\nu) \\
[\mathcal{P}_\mu, \mathcal{K}_\nu] &= 2i (\eta_{\mu\nu} \mathcal{D} + \mathcal{M}_{\mu\nu})
\end{aligned} \tag{4.35}$$

It is also important to note that the six dimensional Poincaré mass operator vanishes identically

$$\mathcal{M}^2 = \eta_{\mu\nu} \mathcal{P}^\mu \mathcal{P}^\nu = 0 \tag{4.36}$$

for the minimal unitary realization of $SO^*(8)$ given above.

In Appendix B, we give the compact 3-grading of $SO^*(8)$ with respect to the maximal compact subgroup $SU(4) \times U(1)_E$:

$$\mathfrak{so}^*(8) = \mathfrak{C}^- \oplus \mathfrak{C}^0 \oplus \mathfrak{C}^+ \tag{4.37}$$

where the $\mathfrak{u}(1)_E$ generator H that determines the compact 3-grading is

$$H = N_0 + \frac{1}{2} (K_+ + K_-) . \tag{4.38}$$

It is the AdS energy operator or the conformal Hamiltonian when $SO^*(8)$ is taken as the seven-dimensional AdS group or the six-dimensional conformal group, respectively. It should also be pointed out that, in the earlier noncompact 3-grading with respect to $\mathfrak{N}^0 = \mathfrak{su}^*(4) \oplus \mathfrak{so}(1, 1)_\mathcal{D}$, this AdS energy corresponds to $\frac{1}{2} (\mathcal{K}_0 + \mathcal{P}_0)$. (See equation (4.34).)

In Appendix B, we also give the decomposition of the Lie subalgebra $\mathfrak{su}(4)$ of \mathfrak{C}^0 with respect to its subalgebra $\mathfrak{su}(2)_S \oplus \mathfrak{su}(2)_A \oplus \mathfrak{u}(1)_J$ where the $U(1)$ charge

$$J = N_0 - \frac{1}{2}(K_+ + K_-) . \quad (4.39)$$

is equal to $\frac{1}{2}(\mathcal{K}_5 - \mathcal{P}_5)$ in the above noncompact 3-grading with respect to $\mathfrak{N}^0 = \mathfrak{su}^*(4) \oplus \mathfrak{so}(1,1)_{\mathcal{D}}$. (See equation (4.34).)

5. Distinguished $SU(1,1)_K$ subgroup of $SO^*(8)$ generated by the isotonic (singular) oscillators

Note that in terms of the oscillators a_m, b_m (and their respective hermitian conjugates a^m, b^m) and the coordinate x and its conjugate momentum p , the $\mathfrak{u}(1)$ generator H (as given in equation (4.38)) has the following form:

$$\begin{aligned} H &= N_0 + \frac{1}{2}(K_+ + K_-) \\ &= \frac{1}{2}(N_a + N_b) + 1 + \frac{1}{4}(x^2 + p^2) + \frac{1}{8x^2} \left(8\mathcal{S}^2 + \frac{3}{2} \right) \\ &= H_a + H_b + H_{\odot} \end{aligned} \quad (5.1)$$

This H plays the role of the seven dimensional AdS energy operator or the six dimensional conformal Hamiltonian. H_a and H_b are the contributions to the Hamiltonian from a -type and b -type oscillators, that correspond to standard non-singular harmonic oscillators. On the other hand, H_{\odot} is the Hamiltonian of a singular harmonic oscillator with a potential function

$$V(x) = \frac{G}{x^2} \quad \text{where} \quad G = \frac{1}{4} \left(8\mathcal{S}^2 + \frac{3}{2} \right) . \quad (5.2)$$

H_{\odot} also arises as the Hamiltonian of conformal quantum mechanics [66] with G playing the role of coupling constant [50]. In some literature it is also referred to as the isotonic oscillator [67, 68].

Let us now consider this singular harmonic oscillator Hamiltonian

$$\begin{aligned} H_{\odot} &= \frac{1}{2}(K_+ + K_-) = \frac{1}{4}(x^2 + p^2) + \frac{1}{8x^2} \left(8\mathcal{S}^2 + \frac{3}{2} \right) \\ &= \frac{1}{4} \left(x^2 - \frac{\partial^2}{\partial x^2} \right) + \frac{1}{8x^2} \left(8\mathcal{S}^2 + \frac{3}{2} \right) . \end{aligned} \quad (5.3)$$

Together with the following generators B_- and B_+ of \mathfrak{C}^- and \mathfrak{C}^+ subspaces of $SO^*(8)$ (see

Appendix B):

$$\begin{aligned}
B_- &= \frac{i}{2} [\Delta + i(K_+ - K_-)] = \frac{1}{4} (x + ip)^2 - \frac{1}{2x^2} \left(2\mathcal{S}^2 + \frac{3}{8} \right) \\
&= \frac{1}{4} \left(x + \frac{\partial}{\partial x} \right)^2 - \frac{1}{2x^2} \left(2\mathcal{S}^2 + \frac{3}{8} \right) \\
B_+ &= -\frac{i}{2} [\Delta - i(K_+ - K_-)] = \frac{1}{4} (x - ip)^2 - \frac{1}{2x^2} \left(2\mathcal{S}^2 + \frac{3}{8} \right) \\
&= \frac{1}{4} \left(x - \frac{\partial}{\partial x} \right)^2 - \frac{1}{2x^2} \left(2\mathcal{S}^2 + \frac{3}{8} \right)
\end{aligned} \tag{5.4}$$

H_\odot generates the distinguished $\mathfrak{su}(1, 1)_K$ subalgebra³

$$[B_-, B_+] = 2H_\odot \quad [H_\odot, B_+] = +B_+ \quad [H_\odot, B_-] = -B_- . \tag{5.5}$$

For a given eigenvalue $\mathfrak{s}(\mathfrak{s} + 1)$ of the quadratic Casimir operator \mathcal{S}^2 of $SU(2)_S$, the wave functions corresponding to the lowest energy eigenvalue of this singular harmonic oscillator Hamiltonian H_\odot will be superpositions of functions of the form $\psi_0^{(\alpha_s)}(x) \Lambda(\mathfrak{s}, m_s)$, where $\Lambda(\mathfrak{s}, m_s)$ is an eigenstate of \mathcal{S}^2 and S_0 , independent of x :

$$\mathcal{S}^2 \Lambda(\mathfrak{s}, m_s) = \mathfrak{s}(\mathfrak{s} + 1) \Lambda(\mathfrak{s}, m_s) \quad S_0 \Lambda(\mathfrak{s}, m_s) = m_s \Lambda(\mathfrak{s}, m_s) \tag{5.6}$$

and $\psi_0^{(\alpha_s)}(x)$ is a function that satisfies

$$B_- \psi_0^{(\alpha_s)}(x) \Lambda(\mathfrak{s}, m_s) = 0 \tag{5.7}$$

whose solution is given by [69]

$$\psi_0^{(\alpha_s)}(x) = C_0 x^{\alpha_s} e^{-x^2/2} \tag{5.8}$$

where C_0 is a normalization constant and

$$\alpha_s = \frac{1}{2} + \sqrt{1 + 4\mathfrak{s}(\mathfrak{s} + 1)} = 2\mathfrak{s} + \frac{3}{2} . \tag{5.9}$$

The normalizability of the state imposes the constraint

$$\alpha_s \geq \frac{1}{2} . \tag{5.10}$$

Clearly, $\psi_0^{(\alpha_s=2\mathfrak{s}+3/2)}(x) \Lambda(\mathfrak{s}, m_s)$ is an eigenstate of H_\odot with eigenvalue $E_{\odot,0}^{(\alpha_s)} = (\mathfrak{s} + 1)$:

$$H_\odot \psi_0^{(2\mathfrak{s}+3/2)}(x) \Lambda(\mathfrak{s}, m_s) = (\mathfrak{s} + 1) \psi_0^{(2\mathfrak{s}+3/2)}(x) \Lambda(\mathfrak{s}, m_s) . \tag{5.11}$$

³This is the $SU(1, 1)$ subgroup generated by the longest root vector.

The lowest energy normalizable eigenstate of H_\odot corresponds to the case $\mathfrak{s} = 0$ (therefore $\alpha_{\mathfrak{s}} = \frac{3}{2}$). Note that $\Lambda(0,0)$ is simply the Fock vacuum $|0\rangle$ of a - and b -type oscillators. This “ground state” has energy

$$E_{\odot,0}^{(3/2)} = 1. \quad (5.12)$$

The higher energy eigenstates of H_\odot can be obtained from $\psi_0^{(3/2)}(x) \Lambda(0,0)$ by acting on it repeatedly with the raising operator B_+ :

$$\psi_n^{(3/2)}(x) \Lambda(0,0) = C_n (B_+)^n \psi_0^{(3/2)}(x) \Lambda(0,0) \quad (5.13)$$

where C_n are normalization constants. They correspond to energy eigenvalues:

$$H_\odot \psi_n^{(3/2)}(x) \Lambda(0,0) = E_{\odot,n}^{(3/2)} \psi_n^{(3/2)}(x) \Lambda(0,0) \quad (5.14)$$

where

$$E_{\odot,n}^{(3/2)} = E_{\odot,0}^{(3/2)} + n = 1 + n. \quad (5.15)$$

We shall denote the corresponding states as $\left| \psi_n^{(3/2)}(x) \Lambda(0,0) \right\rangle = \left| \psi_n^{(3/2)}(x) \right\rangle \otimes |\Lambda(0,0)\rangle$ and refer to them as the particle basis of the state space of the (isotonic) singular oscillator.

6. $SU(2)_S \times SU(2)_A \times U(1)_J$ Basis of the Minimal Unitary Representation of $SO^*(8)$

Consider the tensor product of Fock spaces of the oscillators a^m and b^m . The vacuum state $|0\rangle$ is annihilated by all a_m and b_m :

$$a_m |0\rangle = b_m |0\rangle = 0 \quad (6.1)$$

where $m = 1, 2$. Note that $|\Lambda(0,0)\rangle = |0\rangle$. A “particle basis” of states in this Fock space is provided by tensor products of the following states

$$|n_{a,m}\rangle = \frac{1}{\sqrt{n_{a,m}!}} (a^m)^{n_{a,m}} |0\rangle \quad |n_{b,m}\rangle = \frac{1}{\sqrt{n_{b,m}!}} (b^m)^{n_{b,m}} |0\rangle \quad (6.2)$$

where $m = 1, 2$ and $n_{a,m}$ and $n_{b,m}$ are non-negative integers.

To construct a “particle basis” of the Hilbert space of the minimal unitary representation of $SO^*(8)$, consider the following tensor products of the above states with the state space of the singular (isotonic) oscillator:

$$|n_{a,1}\rangle \otimes |n_{a,2}\rangle \otimes |n_{b,1}\rangle \otimes |n_{b,2}\rangle \otimes \left| \psi_n^{(\alpha_s)}(x) \Lambda(0,0) \right\rangle$$

and denote them as

$$(a^1)^{n_{a,1}} (a^2)^{n_{a,2}} (b^1)^{n_{b,1}} (b^2)^{n_{b,2}} \left| \psi_n^{(\alpha_s)}(x) \right\rangle$$

or simply as

$$\left| \psi_n^{(\alpha_s)}(x) ; n_{a,1}, n_{a,2}, n_{b,1}, n_{b,2} \right\rangle.$$

For a fixed $N = n_{a,1} + n_{a,2} + n_{b,1} + n_{b,2}$, these states transform in the $(\frac{N}{2}, \frac{N}{2})$ representation under the $SU(2)_S \times SU(2)_A$ subgroup:

$$\begin{aligned} \mathfrak{s} &= \frac{N}{2} & \mathfrak{a} &= \frac{N}{2} \\ \mathfrak{s}_3 &= \frac{1}{2}(n_{a,1} + n_{a,2} - n_{b,1} - n_{b,2}) & \mathfrak{a}_3 &= \frac{1}{2}(n_{a,1} - n_{a,2} + n_{b,1} - n_{b,2}) \end{aligned} \quad (6.3)$$

However they are, in general, not eigenstates of $U(1)_J$ or the energy operator H (AdS_7 energy or $6D$ conformal Hamiltonian) that determines the compact 3-grading of $SO^*(8)$, given in appendix B, and commutes with $SU(4)$ generators.

There exists a unique lowest energy state in this Hilbert space, namely

$$\left| \psi_0^{(3/2)}(x) ; 0, 0, 0, 0 \right\rangle \quad (6.4)$$

that is annihilated by the following six operators in \mathfrak{C}^- subspace of $\mathfrak{so}^*(8)$:

$$\begin{aligned} Y_m &= \frac{1}{2} (U_m - i \tilde{U}_m) & N_- &= a_1 b_2 - a_2 b_1 \\ Z_m &= \frac{1}{2} (V_m - i \tilde{V}_m) & B_- &= \frac{i}{2} [\Delta + i(K_+ - K_-)] \end{aligned} \quad (6.5)$$

in the compact three grading, and transforms as a singlet of $SU(2)_S \times SU(2)_A$ and is an eigenstate of H and J with eigenvalue $E = 2$ and $\mathfrak{J} = 0$, respectively. This state is also a singlet of $SU(4)$ subgroup. Hence the minimal unitary representation of $SO^*(8)$ is a unitary lowest weight representation. All the other states of the particle basis of the minrep with higher energies can be obtained from $\left| \psi_0^{(3/2)}(x) ; 0, 0, 0, 0 \right\rangle$ by repeatedly acting on it with the following operators in \mathfrak{C}^+ subspace of $\mathfrak{so}^*(8)$:

$$\begin{aligned} Y^m &= \frac{1}{2} (U^m + i \tilde{U}^m) & N_+ &= a^1 b^2 - a^2 b^1 \\ Z^m &= \frac{1}{2} (V^m + i \tilde{V}^m) & B_+ &= -\frac{i}{2} [\Delta - i(K_+ - K_-)] \end{aligned} \quad (6.6)$$

The above six operators in \mathfrak{C}^+ transform under $\mathfrak{su}(2)_S \oplus \mathfrak{su}(2)_A \oplus \mathfrak{u}(1)_J$ as follows:

$$6 = (1/2, 1/2)^0 \oplus (0, 0)^{+1} \oplus (0, 0)^{-1}.$$

The operators (Y^1, Z^1) and (Y^2, Z^2) form two doublets under $\mathfrak{su}(2)_S$. The operators (Y^1, Y^2) and (Z^1, Z^2) form two doublets under $\mathfrak{su}(2)_A$. N_+ and B_+ are both singlets under $\mathfrak{su}(2)_S$ and $\mathfrak{su}(2)_A$. Y^m and Z^m have zero J -charge, while N_+ and B_+ have J -charges $+1$ and -1 , respectively.

We list the charges of these six operators with respect to (S_0, A_0, J) in Table 1.

Table 1: Charges \mathfrak{s}_3 , \mathfrak{a}_3 and \mathfrak{J} of \mathfrak{C}^+ operators of $\mathfrak{so}^*(8)$ with respect to S_0 , A_0 and J , respectively.

\mathfrak{C}^+ generator	Y^1	Y^2	Z^1	Z^2	N_+	B_+
\mathfrak{s}_3	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0
\mathfrak{a}_3	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0
\mathfrak{J}	0	0	0	0	+1	-1

The \mathfrak{C}^+ operators commute with each other and satisfy the following important relation:

$$Y^1 Z^2 - Y^2 Z^1 = N_+ B_+ \quad (6.7)$$

All the states that belong to a given AdS energy level form an irrep of $SU(4)$. We give the $SU(2)_S \times SU(2)_A \times U(1)_J$ decomposition of these $SU(4)$ irreps in Table 2 for the first three energy levels together with their respective $SU(4)$ Dynkin labels.⁴

Table 2: The $SU(2)_S \times SU(2)_A \times U(1)_J$ content of the first three energy levels of minimal unitary representation of $SO^*(8)$ and their $SU(4)$ Dynkin labels. The irreps are labeled by their spins $(\mathfrak{s}, \mathfrak{a})$ and the J -charge as $(\mathfrak{s}, \mathfrak{a})^{\mathfrak{J}}$. \mathfrak{s} , \mathfrak{a} , \mathfrak{J} and E labels of the lowest energy state $\left| \psi_0^{(3/2)}(x) ; 0, 0, 0, 0 \right\rangle$ uniquely identify the $SO^*(8)$ irrep.

Irrep $(\mathfrak{s}, \mathfrak{a})^{\mathfrak{J}}$	E	$SU(4)_{Dynkin}$
$(0, 0)^0$	2	$(0, 0, 0)$
$(0, 0)^{-1} \oplus (0, 0)^{+1}$ $\oplus \left(\frac{1}{2}, \frac{1}{2}\right)^0$	3	$(0, 1, 0)$

⁴Our convention of Dynkin labels is such that, the fundamental representation $\mathbf{4}$ of $SU(4)$ corresponds to $(1, 0, 0)$.

Table 2: (continued)

Irrep $(\mathfrak{s}, \mathfrak{a})^{\mathfrak{j}}$	E	$SU(4)$ Dynkin
$(0, 0)^{-2} \oplus (0, 0)^0 \oplus (0, 0)^{+2}$ $\oplus \left(\frac{1}{2}, \frac{1}{2}\right)^{-1} \oplus \left(\frac{1}{2}, \frac{1}{2}\right)^{+1}$ $\oplus (1, 1)^0$	4	(0, 2, 0)
\vdots	\vdots	\vdots

By acting on the lowest weight state $|\Omega\rangle$ with \mathfrak{C}^+ generators n times, one obtains a set of states that are eigenstates of H with energy eigenvalues $n+2$. They form an $SU(4)$ irrep with Dynkin labels $(0, n, 0)$, which decomposes into the following irreps of $SU(2)_S \times SU(2)_A \times U(1)_J$ subgroup labelled by $(\mathfrak{s}, \mathfrak{a})^{\mathfrak{j}}$:

$$\begin{aligned}
(0, n, 0)_{SU(4) \text{ Dynkin}} &= (0, 0)^{-n} \oplus (0, 0)^{-n+2} \oplus \dots \oplus (0, 0)^n \\
&\oplus \left(\frac{1}{2}, \frac{1}{2}\right)^{-n+1} \oplus \left(\frac{1}{2}, \frac{1}{2}\right)^{-n+3} \oplus \dots \oplus \left(\frac{1}{2}, \frac{1}{2}\right)^{n-1} \\
&\oplus (1, 1)^{-n+2} \oplus (1, 1)^{-n+4} \oplus \dots \oplus (1, 1)^{n-2} \\
&\vdots \\
&\oplus \left(\frac{n}{2}, \frac{n}{2}\right)^0
\end{aligned} \tag{6.8}$$

Comparing the $SU(4)$ content of the minrep of $SO^*(8)$ with that of the scalar doubleton representation of the seven dimensional AdS group $SO(6, 2) \simeq SO^*(8)$ obtained by the oscillator method [22, 59, 60], we see that they coincide exactly. Thus the minrep of $SO^*(8)$ is simply the scalar doubleton representation of $SO^*(8)$ whose Poincaré limit in AdS_7 is singular, just like Dirac's singletons of $SO(3, 2)$ in AdS_4 . The doubleton representations correspond to massless representations of $SO^*(8)$ considered as six dimensional conformal group [22, 60].

At this point we should stress one important point. In the oscillator construction of the scalar doubleton given in [22], one is working in the Fock space of two sets of oscillators transforming in the fundamental representation of the maximal compact subgroup $U(4)$. The Fock space of all eight oscillators decomposes into an infinite family of doubleton representations that correspond to massless conformal fields of ever increasing spin. In contrast, the minimal unitary representation we constructed above is over the tensor product of Fock space of four oscillators and the state space of a singular oscillator.

In the subsequent sections we shall extend our construction of the minrep of $SO^*(8)$ to the construction of minimal unitary representation of the supergroup $OSp(8^*|2N)$ which correspond to supermultiplets of massless conformal fields in six dimensions.

7. Construction of Finite-Dimensional Representations of $USp(2N)$ in terms of Fermionic Oscillators

We define two sets of N fermionic oscillators α_r, β_r and their hermitian conjugates $\alpha^r = (\alpha_r)^\dagger, \beta^r = (\beta_r)^\dagger$ ($r = 1, 2, \dots, N$), such that they satisfy the usual anti-commutation relations:

$$\{\alpha_r, \alpha^s\} = \{\beta_r, \beta^s\} = \delta_r^s \quad \{\alpha_r, \alpha_s\} = \{\alpha_r, \beta_s\} = \{\beta_r, \beta_s\} = 0 \quad (7.1)$$

The Lie algebra $\mathfrak{usp}(2N)$ has a 3-graded decomposition with respect to its subalgebra $\mathfrak{u}(N)$ as follows:

$$\begin{aligned} \mathfrak{usp}(2N) &= \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} \\ &= S_{rs} \oplus M_s^r \oplus S^{rs} \end{aligned} \quad (7.2)$$

where

$$\begin{aligned} S_{rs} &= \alpha_r \beta_s + \alpha_s \beta_r \\ M_s^r &= \alpha^r \alpha_s - \beta_s \beta^r \\ S^{rs} &= \beta^r \alpha^s + \beta^s \alpha^r = (S_{rs})^\dagger. \end{aligned} \quad (7.3)$$

They satisfy the commutation relations:

$$\begin{aligned} [S_{rs}, S^{tu}] &= -\delta_s^t M_r^u - \delta_r^t M_s^u - \delta_s^u M_r^t - \delta_r^u M_s^t \\ [M_s^r, S_{tu}] &= -\delta_u^r S_{st} - \delta_t^r S_{su} \\ [M_s^r, S^{tu}] &= \delta_s^u S^{rt} + \delta_s^t S^{ru} \\ [M_s^r, M_u^t] &= \delta_s^t M_u^r - \delta_u^r M_s^t \end{aligned} \quad (7.4)$$

The quadratic Casimir of $\mathfrak{usp}(2N)$ is given by

$$\begin{aligned} \mathcal{C}_2[\mathfrak{usp}(2N)] &= M_s^r M_r^s + \frac{1}{2} (S_{rs} S^{rs} + S^{rs} S_{rs}) \\ &= N(N+2) - (N_\alpha + N_\beta) [(N_\alpha + N_\beta) + 2] - 8 \alpha^{(r} \beta^{s)} \alpha_{(r} \beta_{s)} \end{aligned} \quad (7.5)$$

where “ (rs) ” represents symmetrization of weight one, $\alpha_{(r} \beta_{s)} = \frac{1}{2} (\alpha_r \beta_s + \alpha_s \beta_r)$.

We choose the fermionic Fock vacuum such that

$$\alpha_r |0\rangle = \beta_r |0\rangle = 0. \quad (7.6)$$

To generate an irrep of $USp(2N)$ in the Fock space in a $U(N)$ basis, one chooses a set of states $|\Omega\rangle$, transforming irreducibly under $U(N)$ and are annihilated by all grade -1 operators S_{rs} (i.e. $S_{rs} |\Omega\rangle = 0$), and act on them with grade $+1$ operators S^{rs} [63]:

$$\{|\Omega\rangle, S^{rs} |\Omega\rangle, S^{rs} S^{tu} |\Omega\rangle, \dots\} \quad (7.7)$$

The possible sets of states $|\Omega\rangle$, that transform irreducibly under $U(N)$ and are annihilated by S_{rs} , are of the form

$$\alpha^{r_1} \alpha^{r_2} \dots \alpha^{r_m} |0\rangle \quad (7.8)$$

or of the equivalent form

$$\beta^{r_1} \beta^{r_2} \dots \beta^{r_m} |0\rangle \quad (7.9)$$

where $m \leq N$. They lead to irreps of $USp(2N)$ with Dynkin labels [63]

$$\left(\underbrace{0, \dots, 0}_{(N-m-1)}, 1, \underbrace{0, \dots, 0}_{(m)} \right). \quad (7.10)$$

In addition, we have the following states

$$\alpha^{[r} \beta^{s]} |0\rangle = \frac{1}{2} (\alpha^r \beta^s - \alpha^s \beta^r) |0\rangle \quad (7.11)$$

that are annihilated by all grade -1 operators S_{tu} . They lead to the irrep of $USp(2N)$ with Dynkin labels

$$\left(\underbrace{0, \dots, 0}_{(N-3)}, 1, 0, 0 \right). \quad (7.12)$$

Note that in the special case of $\mathfrak{usp}(4)$, the states $\alpha^r \alpha^s |0\rangle$, $\beta^r \beta^s |0\rangle$ and $\alpha^{[r} \beta^{s]} |0\rangle$ all lead to the trivial representation.

It is important to also note that the following bilinears of fermionic oscillators

$$F_+ = \alpha^r \beta_r \quad F_- = \beta^r \alpha_r \quad F_0 = \frac{1}{2} (N_\alpha - N_\beta) \quad (7.13)$$

where $N_\alpha = \alpha^r \alpha_r$ and $N_\beta = \beta^r \beta_r$ are the respective number operators, generate a $\mathfrak{usp}(2)_F \simeq \mathfrak{su}(2)_F$ algebra

$$[F_+, F_-] = 2 F_0 \quad [F_0, F_\pm] = \pm F_\pm \quad (7.14)$$

that commutes with the $\mathfrak{usp}(2N)$ algebra defined above. Nonetheless, the equivalent irreps of $USp(2N)$ constructed from the states $|\Omega\rangle$ involving only α -type excitations or β -type excitations can form non-trivial representations of this $USp(2)_F$.

For example, the two irreps with Dynkin labels $(1,0)$ of $USp(4)$ constructed from $\alpha^r |0\rangle$ and $\beta^r |0\rangle$ form spin $\frac{1}{2}$ representation (doublet) of $USp(2)_F$. The three singlet irreps of $USp(4)$ corresponding to $\alpha^r \alpha^s |0\rangle$, $\beta^r \beta^s |0\rangle$ and $\alpha^{[r} \beta^{s]} |0\rangle$ form the spin 1 representation (triplet) of $USp(2)_F$. The irrep of $USp(4)$ with Dynkin labels $(0,1)$ defined by the vacuum state $|\Omega\rangle = |0\rangle$ is a singlet of $USp(2)_F$.

We should note that the representations of $USp(2N)$ obtained above by using two sets of fermionic oscillators transforming in the fundamental representation of the subgroup $U(N)$ are the compact analogs of the doubleton representations of $SO^*(2M)$ constructed using two sets of bosonic oscillators transforming in the fundamental representation of $U(M)$ [63]. By realizing the generators of $USp(2N)$ in terms of an arbitrary (even) number of sets of oscillators, one can construct all the finite dimensional representations of $USp(2N)$ [22, 63].

8. Minimal Unitary Representations of Supergroups $OSp(8^*|2N)$ with Even Subgroups $SO^*(8) \times USp(2N)$

The noncompact groups $SO^*(2M)$ with maximal compact subgroups $U(M)$ have extensions to supergroups $OSp(2M^*|2N)$ with the even subgroups $SO^*(2M) \times USp(2N)$ that admit positive energy unitary representations. In this section we shall study the minimal unitary representations of $OSp(8^*|2N)$, leaving to a separate study the minimal representations of $OSp(2M^*|2N)$ for arbitrary M [70]. We define minimal unitary representations (supermultiplets) of $OSp(2M^*|2N)$ as those irreducible unitary representations (supermultiplets) that contain the minimal unitary representation of $SO^*(2M)$.

The superalgebra $\mathfrak{osp}(8^*|2N)$ has a 5-grading

$$\mathfrak{osp}(8^*|2N) = \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)} \quad (8.1)$$

with respect to the subsuperalgebra

$$\mathfrak{g}^{(0)} = \mathfrak{osp}(4^*|2N) \oplus \mathfrak{su}(2) \oplus \mathfrak{so}(1,1)_\Delta. \quad (8.2)$$

In the extension of the minimal unitary realization of $SO^*(8)$ to that of $OSp(8^*|2N)$, the grade -2 generator K_- remains unchanged. However, now grade -1 subspace contains both even (bosonic) and odd (fermionic) generators. More precisely, the grade -1 subspace $\mathfrak{g}^{(-1)}$ of $\mathfrak{osp}(8^*|2N)$ contains 8 bosonic generators:

$$\begin{aligned} U_m &= x a_m & U^m &= x a^m \\ V_m &= x b_m & V^m &= x b^m \end{aligned} \quad (8.3)$$

and $4N$ supersymmetry generators:

$$\begin{aligned} Q_r &= x \alpha_r & Q^r &= x \alpha^r \\ S_r &= x \beta_r & S^r &= x \beta^r \end{aligned} \quad (8.4)$$

They (anti-)commute into the single bosonic generator K_- in grade -2 subspace $\mathfrak{g}^{(-2)}$ as follows:

$$\begin{aligned} [U_m, U^n] &= [V_m, V^n] = 2 \delta_m^n K_- \\ \{Q_r, Q^s\} &= \{S_r, S^s\} = 2 \delta_r^s K_- \end{aligned} \quad (8.5)$$

Even and odd generators in $\mathfrak{g}^{(-1)}$ commute with each other and together form a super Heisenberg algebra with

$$K_- = \frac{1}{2} x^2$$

as its “central charge.”

Now the $\mathfrak{su}(2)_S$ subalgebra in grade zero subspace $\mathfrak{g}^{(0)}$ of $\mathfrak{so}^*(8)$ receives contributions from fermionic oscillators in the supersymmetric extension of $\mathfrak{so}^*(8)$ to $\mathfrak{osp}(8^*|2N)$. The

resultant $\mathfrak{su}(2)$ that commutes with $\mathfrak{osp}(4^*|2N)$ of grade zero subalgebra is simply the diagonal subalgebra of $\mathfrak{su}(2)_S$ and $\mathfrak{su}(2)_F$ defined earlier in equation (7.13). We shall label it as $\mathfrak{su}(2)_T$ in the supersymmetric extension. Its generators are:

$$\begin{aligned} T_+ &= S_+ + F_+ = a^m b_m + \alpha^r \beta_r \\ T_- &= S_- + F_- = b^m a_m + \beta^r \alpha_r \\ T_0 &= S_0 + F_0 = \frac{1}{2} (N_a - N_b + N_\alpha - N_\beta) \end{aligned} \quad (8.6)$$

so that

$$[T_+, T_-] = 2T_0 \quad [T_0, T_\pm] = \pm T_\pm. \quad (8.7)$$

The subsuperalgebra $\mathfrak{osp}(4^*|2N)$ belonging to grade zero subspace $\mathfrak{g}^{(0)}$ has an even subalgebra $\mathfrak{so}^*(4) \oplus \mathfrak{usp}(2N)$. The generators of $\mathfrak{so}^*(4) = \mathfrak{su}(2)_A \oplus \mathfrak{su}(1,1)_N$ were denoted as $A_{\pm,0}$ and $N_{\pm,0}$ (see equation (4.10)), and the generators of $\mathfrak{usp}(2N)$ were denoted as S_{rs} , M^r_s , S^{rs} ($r, s, \dots = 1, \dots, N$) (see equation (7.3)).

The $8N$ supersymmetry generators of $\mathfrak{osp}(4^*|2N)$ are realized by the following bilinears:

$$\begin{aligned} \Pi_{mr} &= a_m \beta_r - b_m \alpha_r & \overline{\Pi}^{mr} &= (\Pi_{mr})^\dagger = a^m \beta^r - b^m \alpha^r \\ \Sigma_m^r &= a_m \alpha^r + b_m \beta^r & \overline{\Sigma}_r^m &= (\Sigma_m^r)^\dagger = a^m \alpha_r + b^m \beta_r \end{aligned} \quad (8.8)$$

Recalling that

$$\mathcal{C}_2 [\mathfrak{su}(2)_A] = \mathcal{C}_2 [\mathfrak{su}(1,1)_N] = \mathcal{C}_2 [\mathfrak{su}(2)_S] \quad (8.9)$$

we find that the quadratic Casimir of $\mathfrak{osp}(4^*|2N)$ can be written as

$$\mathcal{C}_2 [\mathfrak{osp}(4^*|2N)] = \mathcal{C}_2 [\mathfrak{su}(2)_S] - \frac{1}{8} \mathcal{C}_2 [\mathfrak{usp}(2N)] + \frac{1}{4} \mathcal{F}(\Pi, \Sigma) \quad (8.10)$$

where

$$\mathcal{F}(\Pi, \Sigma) = \Pi_{mr} \overline{\Pi}^{mr} - \overline{\Pi}^{mr} \Pi_{mr} + \Sigma_m^r \overline{\Sigma}_r^m - \overline{\Sigma}_r^m \Sigma_m^r. \quad (8.11)$$

Remarkably, it reduces to the quadratic Casimir of $SU(2)_T$ modulo an additive constant for the minimal unitary realization

$$\mathcal{C}_2 [\mathfrak{osp}(4^*|2N)] = \mathcal{C}_2 [\mathfrak{su}(2)_T] - \frac{N(N-4)}{16} \quad (8.12)$$

where

$$\mathcal{C}_2 [\mathfrak{su}(2)_T] = T_0^2 + \frac{1}{2} (T_+ T_- + T_- T_+) \equiv \mathcal{T}^2. \quad (8.13)$$

Using this result, one can write the grade +2 generator in full generality as

$$K_+ = \frac{1}{2} p^2 + \frac{1}{4x^2} \left(8\mathcal{T}^2 + \frac{3}{2} \right) \quad (8.14)$$

that is valid for all N . The generators in grade +1 subspace $\mathfrak{g}^{(+1)}$ are obtained from the commutators of $\mathfrak{g}^{(-1)}$ generators with K_+ :

$$\begin{aligned}
\tilde{U}_m &= i[U_m, K_+] & \tilde{U}^m &= i[U^m, K_+] \\
\tilde{V}_m &= i[V_m, K_+] & \tilde{V}^m &= i[V^m, K_+] \\
\tilde{Q}_r &= i[Q_r, K_+] & \tilde{Q}^r &= i[Q^r, K_+] \\
\tilde{S}_r &= i[S_r, K_+] & \tilde{S}^r &= i[S^r, K_+]
\end{aligned} \tag{8.15}$$

The explicit form of the even and odd generators of $\mathfrak{g}^{(+1)}$ are as follows:

$$\begin{aligned}
\tilde{U}_m &= -p a_m + \frac{2i}{x} \left[\left(T_0 + \frac{3}{4} \right) a_m + T_- b_m \right] \\
\tilde{U}^m &= -p a^m - \frac{2i}{x} \left[\left(T_0 - \frac{3}{4} \right) a^m + T_+ b^m \right] \\
\tilde{V}_m &= -p b_m - \frac{2i}{x} \left[\left(T_0 - \frac{3}{4} \right) b_m - T_+ a_m \right] \\
\tilde{V}^m &= -p b^m + \frac{2i}{x} \left[\left(T_0 + \frac{3}{4} \right) b^m - T_- a^m \right]
\end{aligned} \tag{8.16}$$

$$\begin{aligned}
\tilde{Q}_r &= -p \alpha_r + \frac{2i}{x} \left[\left(T_0 + \frac{3}{4} \right) \alpha_r + T_- \beta_r \right] \\
\tilde{Q}^r &= -p \alpha^r - \frac{2i}{x} \left[\left(T_0 - \frac{3}{4} \right) \alpha^r + T_+ \beta^r \right] \\
\tilde{S}_r &= -p \beta_r - \frac{2i}{x} \left[\left(T_0 - \frac{3}{4} \right) \beta_r - T_+ \alpha_r \right] \\
\tilde{S}^r &= -p \beta^r + \frac{2i}{x} \left[\left(T_0 + \frac{3}{4} \right) \beta^r - T_- \alpha^r \right]
\end{aligned} \tag{8.17}$$

They form a (super-)Heisenberg algebra together with K_+ :

$$\begin{aligned}
[\tilde{U}_m, \tilde{U}^n] &= [\tilde{V}_m, \tilde{V}^n] = 2 \delta_m^n K_+ \\
\{\tilde{Q}_r, \tilde{Q}^s\} &= \{\tilde{S}_r, \tilde{S}^s\} = 2 \delta_r^s K_+
\end{aligned} \tag{8.18}$$

The commutation relations of grade +1 generators with the grade -2 generator K_- are:

$$\begin{aligned}
[\tilde{U}_m, K_-] &= i U_m & [\tilde{U}^m, K_-] &= i U^m \\
[\tilde{V}_m, K_-] &= i V_m & [\tilde{V}^m, K_-] &= i V^m \\
[\tilde{Q}_r, K_-] &= i Q_r & [\tilde{Q}^r, K_-] &= i Q^r \\
[\tilde{S}_r, K_-] &= i S_r & [\tilde{S}^r, K_-] &= i S^r
\end{aligned} \tag{8.19}$$

In terms of the generators defined above the 5-graded decomposition of the Lie superalgebra $\mathfrak{osp}(8^*|2N)$, defined by the generator Δ , takes the form:

$$\begin{aligned}\mathfrak{osp}(8^*|4) &= \mathfrak{g}^{(-2)} \oplus \mathfrak{g}^{(-1)} \oplus [\mathfrak{osp}(4^*|2N) \oplus \mathfrak{su}(2) \oplus \mathfrak{so}(1,1)_\Delta] \oplus \mathfrak{g}^{(+1)} \oplus \mathfrak{g}^{(+2)} \\ &= K_- \oplus [U_m, U^m, V_m, V^m, Q_r, Q^r, S_r, S^r] \\ &\quad \oplus [A_{\pm,0}, N_{\pm,0}, S_{rs}, M_s^r, S^{rs}, \Pi_{mr}, \bar{\Pi}^{mr}, \Sigma_m^r, \bar{\Sigma}_r^m, T_{\pm,0}, \Delta] \\ &\quad \oplus [\tilde{U}_m, \tilde{U}^m, \tilde{V}_m, \tilde{V}^m, \tilde{Q}_r, \tilde{Q}^r, \tilde{S}_r, \tilde{S}^r] \oplus K_+\end{aligned}\tag{8.20}$$

We give the additional (super-)commutation relations of this superalgebra in the 5-graded basis in appendix D.

9. Compact 3-Grading of $OSp(8^*|2N)$ and its Minimal Unitary Representation

The Lie superalgebra $\mathfrak{osp}(8^*|2N)$ can be given a 3-graded decomposition with respect to its compact subsuperalgebra $\mathfrak{u}(4|N) = \mathfrak{su}(4|N) \oplus \mathfrak{u}(1)_\mathcal{H}$

$$\mathfrak{osp}(8^*|2N) = \mathfrak{C}^- \oplus \mathfrak{C}^0 \oplus \mathfrak{C}^+ \tag{9.1}$$

where

$$\begin{aligned}\mathfrak{C}^- &= \frac{1}{2} (U_m - i\tilde{U}_m) \oplus \frac{1}{2} (V_m - i\tilde{V}_m) \oplus N_- \oplus \frac{i}{2} [\Delta + i(K_+ - K_-)] \oplus S_{rs} \\ &\quad \oplus \frac{1}{2} (Q_r - i\tilde{Q}_r) \oplus \frac{1}{2} (S_r - i\tilde{S}_r) \oplus \Pi_{mr} \\ \mathfrak{C}^0 &= [T_{\pm,0} \oplus A_{\pm,0} \oplus [N_0 - \frac{1}{2}(K_+ + K_-)]] \oplus \frac{1}{2} (U_m + i\tilde{U}_m) \oplus \frac{1}{2} (U^m - i\tilde{U}^m) \\ &\quad \oplus \frac{1}{2} (V_m + i\tilde{V}_m) \oplus \frac{1}{2} (V^m - i\tilde{V}^m) \oplus M_s^r \oplus \left[\frac{1}{2} (K_+ + K_-) + \frac{2}{N} M_0 \right] \oplus \mathcal{H} \\ &\quad \oplus \frac{1}{2} (Q_r + i\tilde{Q}_r) \oplus \frac{1}{2} (Q^r - i\tilde{Q}^r) \oplus \frac{1}{2} (S_r + i\tilde{S}_r) \oplus \frac{1}{2} (S^r - i\tilde{S}^r) \oplus \Sigma_m^r \oplus \bar{\Sigma}_r^m \\ \mathfrak{C}^+ &= \frac{1}{2} (U^m + i\tilde{U}^m) \oplus \frac{1}{2} (V^m + i\tilde{V}^m) \oplus N_+ \oplus -\frac{i}{2} [\Delta - i(K_+ - K_-)] \oplus S^{rs} \\ &\quad \oplus \frac{1}{2} (Q^r + i\tilde{Q}^r) \oplus \frac{1}{2} (S^r + i\tilde{S}^r) \oplus \bar{\Pi}^{mr}\end{aligned}\tag{9.2}$$

The $U(1)$ generator \mathcal{H} that defines the compact 3-grading of $\mathfrak{osp}(8^*|2N)$ is given by

$$\mathcal{H} = \frac{1}{2} (K_+ + K_-) + N_0 + M_0 \tag{9.3}$$

where M_0 is the generator of the $U(1)$ factor of $U(N)$ subgroup of $USp(2N)$ (equation (7.3)):

$$M_0 = \frac{1}{2} (N_\alpha + N_\beta - N) = \frac{1}{2} (N_\alpha - N_\beta) \tag{9.4}$$

Therefore

$$\mathcal{H} = \frac{1}{4} (x^2 + p^2) + \frac{1}{8x^2} \left(8\mathcal{T}^2 + \frac{3}{2} \right) + \frac{1}{2} (N_a + N_b + N_\alpha + N_\beta) + \frac{2-N}{2}. \quad (9.5)$$

plays the role of the “total energy” operator.

Note that, in the supersymmetric extension, the $\mathfrak{u}(1)$ generator H in $\mathfrak{so}^*(8)$ (equation (B.2)), which is the AdS_7 energy, that determines its compact 3-grading becomes

$$\begin{aligned} H_B &= \frac{1}{2} (K_+ + K_-) + N_0 \\ &= \frac{1}{4} (x^2 + p^2) + \frac{1}{8x^2} \left(8\mathcal{T}^2 + \frac{3}{2} \right) + \frac{1}{2} (N_a + N_b) + 1 \\ &= H_\odot + H_a + H_b. \end{aligned} \quad (9.6)$$

The Hamiltonian of the singular oscillator now has contributions from the fermionic oscillators:

$$H_\odot = \frac{1}{4} (x^2 + p^2) + \frac{1}{8x^2} \left(8\mathcal{T}^2 + \frac{3}{2} \right) \quad (9.7)$$

where \mathcal{T}^2 is the quadratic Casimir of $SU(2)_T$, the diagonal subgroup of $SU(2)_S$ and $SU(2)_F$ which are realized in terms of purely bosonic and purely fermionic oscillators, respectively. H_a and H_b remain unchanged in the supersymmetric extension:

$$H_a = \frac{1}{2} (N_a + 1) \quad H_b = \frac{1}{2} (N_b + 1)$$

The explicit expressions for the bosonic operators that belong to the subspace \mathfrak{C}^- of $\mathfrak{osp}(8^*|2N)$ in the compact 3-grading are as follows⁵:

$$\begin{aligned} Y_m &= \frac{1}{2} (U_m - i\tilde{U}_m) = \frac{1}{2} (x + ip) a_m + \frac{1}{x} \left[\left(T_0 + \frac{3}{4} \right) a_m + T_- b_m \right] \\ Z_m &= \frac{1}{2} (V_m - i\tilde{V}_m) = \frac{1}{2} (x + ip) b_m - \frac{1}{x} \left[\left(T_0 - \frac{3}{4} \right) b_m - T_+ a_m \right] \\ N_- &= a_1 b_2 - a_2 b_1 \\ B_- &= \frac{i}{2} [\Delta + i(K_+ - K_-)] = \frac{1}{4} (x + ip)^2 - \frac{1}{8x^2} \left(8\mathcal{T}^2 + \frac{3}{2} \right) \\ S_{rs} &= \alpha_r \beta_s + \alpha_s \beta_r \end{aligned} \quad (9.8)$$

The $4N$ supersymmetry generators in \mathfrak{C}^- subspace are given by:

$$\begin{aligned} \mathfrak{Q}_r &= \frac{1}{2} (Q_r - i\tilde{Q}_r) = \frac{1}{2} (x + ip) \alpha_r + \frac{1}{x} \left[\left(T_0 + \frac{3}{4} \right) \alpha_r + T_- \beta_r \right] \\ \mathfrak{S}_r &= \frac{1}{2} (S_r - i\tilde{S}_r) = \frac{1}{2} (x + ip) \beta_r - \frac{1}{x} \left[\left(T_0 - \frac{3}{4} \right) \beta_r - T_+ \alpha_r \right] \\ \Pi_{mr} &= a_m \beta_r - b_m \alpha_r \end{aligned} \quad (9.9)$$

⁵Note that we are using the same symbols for the generators of $SO^*(8)$ considered as a subgroup of $OSp(8^*|2N)$ that now includes contributions from the fermions.

The operators that belong to \mathfrak{E}^+ subspace are the Hermitian conjugates of those in \mathfrak{E}^- . The bosonic operators in \mathfrak{E}^+ are:

$$\begin{aligned}
Y^m &= \frac{1}{2} (U^m + i \tilde{U}^m) = \frac{1}{2} (x - i p) a^m + \frac{1}{x} \left[\left(T_0 - \frac{3}{4} \right) a^m + T_+ b^m \right] \\
Z^m &= \frac{1}{2} (V^m + i \tilde{V}^m) = \frac{1}{2} (x - i p) b^m - \frac{1}{x} \left[\left(T_0 + \frac{3}{4} \right) b^m - T_- a^m \right] \\
N_+ &= a^1 b^2 - a^2 b^1 \\
B_+ &= -\frac{i}{2} [\Delta - i (K_+ - K_-)] = \frac{1}{4} (x - i p)^2 - \frac{1}{8 x^2} \left(8 \mathcal{T}^2 + \frac{3}{2} \right) \\
S^{rs} &= \alpha^r \beta^s + \alpha^s \beta^r
\end{aligned} \tag{9.10}$$

The $4N$ supersymmetry generators in \mathfrak{E}^+ subspace are:

$$\begin{aligned}
\Omega^r &= \frac{1}{2} (Q^r + i \tilde{Q}^r) = \frac{1}{2} (x - i p) \alpha^r + \frac{1}{x} \left[\left(T_0 - \frac{3}{4} \right) \alpha^r + T_+ \beta^r \right] \\
\mathfrak{S}^r &= \frac{1}{2} (S^r + i \tilde{S}^r) = \frac{1}{2} (x - i p) \beta^r - \frac{1}{x} \left[\left(T_0 + \frac{3}{4} \right) \beta^r - T_- \alpha^r \right] \\
\overline{\Pi}^{mr} &= a^m \beta^r - b^m \alpha^r
\end{aligned} \tag{9.11}$$

We find again the following relation

$$Y^1 Z^2 - Y^2 Z^1 = N_+ B_+ \tag{9.12}$$

among the generators in \mathfrak{E}^+ within the supersymmetric extension of the minrep. We give the (super-)commutation relations between these \mathfrak{E}^- and \mathfrak{E}^+ operators and the explicit form of the generators of grade zero subspace \mathfrak{E}^0 and their (super-)commutation relations in appendix E.

In the supersymmetric extension, the quadratic Casimirs of two $SU(2)$ subgroups are no longer identical. The generators of $SU(2)_A$ remain unchanged, but $SU(2)_S$ generators get contributions from fermions and go over to $SU(2)_T$. (See appendix E for their explicit forms.)

As we showed in section 6, the minimal unitary representation of $SO^*(8) \simeq SO(6, 2)$ is a lowest weight representation with a unique lowest weight vector $|\psi_0^{(3/2)}(x); 0, 0, 0, 0\rangle$ that is annihilated by all the operators in \mathfrak{E}^- subspace and corresponds to a conformal scalar in six dimensions. The lowest weight vector $|\psi_0^{(3/2)}(x); 0, 0, 0, 0\rangle$ is a singlet of the semi-simple part of the little group, namely $SO(4) = SU(2)_S \times SU(2)_A$, of massless states in six dimensions. Now the minimal unitary representation of $OSp(8^*|2N)$ constructed above restricts to a finite number of inequivalent unitary irreducible representations of $SO^*(8)$, whose realization involves fermionic as well as bosonic oscillators. We shall refer to the resulting representations of $SO^*(8)$ as “deformations” of the minimal unitary representation. These deformations of the minimal unitary representation of $SO^*(8)$ also satisfy the Poincaré massless condition

$$\mathcal{M}^2 = \eta_{\mu\nu} P^\mu P^\nu = 0 \tag{9.13}$$

and hence correspond to massless conformal fields in six dimensions. Note that $SO(4)$ is the six dimensional analog of the little group $SO(2)$ of massless states in four dimensions. The minimal unitary representation of $4D$ conformal group $SO(4, 2) = SU(2, 2)/\mathbf{Z}_2$ corresponds to a massless conformal field in four dimensions [55], and its deformations, which are labeled by a real parameter ζ , also describe massless conformal fields. For physical fields, this parameter is simply twice the helicity of a massless unitary representation of the Poincaré subgroup of $SO(4, 2)$. They are the doubleton representations of $SU(2, 2)$, whose supersymmetric extensions were studied in [20, 56, 57]. It was shown long time ago that the corresponding representations of the conformal group $SU(2, 2)$ remain irreducible under the restriction to the four dimensional Poincaré subgroup [58].⁶ We expect the massless doubleton representations of $SO^*(8)$ to remain irreducible under restriction to $6D$ Poincaré subgroup as well.

10. Minimal Unitary Supermultiplet of $\mathfrak{osp}(8^*|2N)$

Since the subgroup $SU(2)_S$ of $SO^*(8)$ is replaced by $SU(2)_T$ when $SO^*(8)$ is extended to the supergroup $OSp(8^*|2N)$, the parameter α in the wave functions $\psi_n^{(\alpha)}(x)$ (as defined in equation (5.8)) now depends on \mathfrak{t} , instead of \mathfrak{s} , where \mathfrak{t} is the $SU(2)_T$ spin.

In this section, for the sake of simplicity, we shall denote the tensor product of the lowest energy state of the “singular” part H_\odot of the bosonic Hamiltonian H with the coordinate wave function

$$\psi_0^{(\alpha_{\mathfrak{t}=3/2})}(x) = C_0 x^{\frac{3}{2}} e^{-x^2/2} \quad (10.1)$$

and the vacuum state of all the bosonic and fermionic oscillators a^m, b^m, α^r and β^r simply as $|\psi_0^{(3/2)}\rangle$:

$$\begin{aligned} a_m |\psi_0^{(3/2)}\rangle &= b_m |\psi_0^{(3/2)}\rangle = 0 \\ \alpha_\mu |\psi_0^{(3/2)}\rangle &= \beta_\mu |\psi_0^{(3/2)}\rangle = 0 \end{aligned} \quad (10.2)$$

Note that for a general state involving bosonic and fermionic excitations,

$$\alpha_{\mathfrak{t}} = 2\mathfrak{t} + 3/2 \quad (10.3)$$

if $\mathfrak{t}(\mathfrak{t} + 1)$ is the eigenvalue of the quadratic Casimir \mathcal{T}^2 of $SU(2)_T$ on that state.

10.1 Minimal unitary supermultiplet of $\mathfrak{osp}(8^*|4)$

First we shall present the results for the case $N = 2$ (i.e. for $USp(4)$), which is relevant to the symmetry supergroup of the S^4 compactification of the eleven dimensional supergravity. The oscillator construction of the unitary supermultiplets of $OSp(8^*|4)$ has been studied in [22, 59, 60]. It has 32 supersymmetry generators, 16 of which belong to grade zero subspace \mathfrak{C}^0 and 8 each belong to grade ± 1 subspaces \mathfrak{C}^\pm .

⁶These representations are sometimes referred to as the ladder representations in the literature.

The state $|\psi_0^{(3/2)}\rangle$ is the unique normalizable lowest energy state annihilated by all 9 bosonic operators as well as all 8 supersymmetry generators in \mathfrak{C}^- subspace. It is a singlet of $SU(4|2)$ subalgebra. By acting on it with grade +1 operators in the subspace \mathfrak{C}^+ , one obtains an infinite set of states which forms a basis for the minimal unitary irreducible representation of $\mathfrak{osp}(8^*|4)$. This infinite set of states can be decomposed into a finite number of irreducible representations of the even subgroup $SO^*(8) \times USp(4)$, with each irrep of $SO^*(8) \simeq SO(6,2)$ corresponding to a massless conformal field in six dimensions.

In Table 3, we present the supermultiplet that is obtained by starting from this unique lowest weight vector

$$|\psi_0^{(3/2)}\rangle \quad (10.4)$$

and acting on it with the generators of grade +1 subspace \mathfrak{C}^+ .

The resulting minimal unitary supermultiplet is the ultra-short doubleton supermultiplet of AdS_7 supergroup $OSp(8^*|4)$ which does not have a Poincaré limit in seven dimensions and whose field theory lives on the boundary of AdS_7 on which $SO^*(8)$ acts as the conformal group [22]. It describes a massless $(2,0)$ conformal supermultiplet whose interacting field theory is believed to be dual to M-theory on $AdS_7 \times S^4$ [23]. The corresponding minimal supermultiplet of 4D superconformal algebra $SU(2,2|4)$ is the $\mathcal{N} = 4$ Yang-Mills supermultiplet in four dimensions [55]. In earlier literature, it was called the CPT self-conjugate doubleton supermultiplet. In the twistorial oscillator approach, the lowest weight vector $|\Omega\rangle$ for this supermultiplet is the vacuum vector $|0\rangle$ of all the oscillators in the $SU(4|2) \times U(1)$ basis [22, 59].

Recalling that the positive energy unitary irreducible representations of $SO^*(8)$ are uniquely labeled by their lowest energy $SU(4)$ irreps, we note that each such $SU(4)$ irrep can in turn be uniquely labeled by an irrep of its subgroup $SU(2)_T \times SU(2)_A \times U(1)_J$ with respect to which it admits a compact three grading. Denoting the $SU(2)_T \times SU(2)_A$ spins as $\mathfrak{t}, \mathfrak{a}$ and the eigenvalue of J as \mathfrak{J} , the Table 3 also gives the decompositions of the lowest energy $SU(4)$ irreps of $SO^*(8)$ in the minimal supermultiplet. The $USp(4)$ transformation properties of these $SO^*(8)$ irreps follow from the results of section 7.

In Table 3, $(\mathcal{Q})^n |\Omega\rangle$ denotes symbolically a lowest energy irrep of $SO^*(8)$ obtained by acting on the lowest weight state $|\Omega\rangle$ with n copies of supersymmetry generators $\mathcal{Q} = \{\mathfrak{Q}^r, \mathfrak{S}^r, \Pi^{mr}\}$.

Table 3: The minimal unitary supermultiplet of $\mathfrak{osp}(8^*|4)$ defined by the lowest weight vector $|\psi_0^{(3/2)}\rangle$. The decomposition of $SU(4)$ irreps with respect to $SU(2)_T \times SU(2)_A \times U(1)_J$ is denoted by $(\mathfrak{t}, \mathfrak{a})^{\mathfrak{j}}$. H is the AdS energy (negative conformal dimension), and \mathcal{H} is the total energy. The Dynkin labels of the lowest energy $SU(4)$ representations of $SO^*(8)$ coincide with the Dynkin labels of the corresponding massless $6D$ conformal fields under the Lorentz group $SU^*(4)$. $USp(4)$ Dynkin labels of these fields are also given.

States	H	\mathcal{H}	$(\mathfrak{t}, \mathfrak{a})^{\mathfrak{j}}$	$SU(4) = SU^*(4)$ Dynkin	$USp(4)$ Dynkin
$ \psi_0^{(3/2)}\rangle$	2	1	$(0, 0)^0$	(0,0,0)	(0,1)
$\mathcal{Q} \psi_0^{(3/2)}\rangle$	$\frac{5}{2}$	2	$(\frac{1}{2}, 0)^{-\frac{1}{2}} \oplus (0, \frac{1}{2})^{+\frac{1}{2}}$	(1,0,0)	(1,0)
$(\mathcal{Q})^2 \psi_0^{(3/2)}\rangle$	3	3	$(1, 0)^{-1} \oplus (\frac{1}{2}, \frac{1}{2})^0 \oplus (0, 1)^{+1}$	(2,0,0)	(0,0)

10.2 Minimal unitary supermultiplet of $OSp(8^*|2N)$

The supergroup $OSp(8^*|2N)$ has $16N$ supersymmetry generators, $8N$ of which belong to grade zero subspace \mathfrak{E}^0 and $4N$ each belong to grade ± 1 subspaces \mathfrak{E}^\pm in the compact three grading. Once again, the state $|\psi_0^{(3/2)}\rangle$ is the unique normalizable lowest energy state annihilated by all $6 + N(N+1)/2$ bosonic operators as well as all $4N$ supersymmetry generators in \mathfrak{E}^- subspace. It is a singlet of the subsuperalgebra $\mathfrak{su}(4|N)$. By acting on it with grade $+1$ operators in the subspace \mathfrak{E}^+ , one obtains an infinite set of states which forms a basis for the minimal unitary irreducible representation of $\mathfrak{osp}(8^*|2N)$. This infinite set of states can be decomposed into a finite number of irreducible representations of the even subgroup $SO^*(8) \times USp(2N)$, with each irrep of $SO^*(8) \simeq SO(6, 2)$ corresponding to a massless conformal field in six dimensions.

In Table 4, we present the minimal unitary supermultiplet of $\mathfrak{osp}(8^*|2N)$ obtained by starting from the lowest weight state

$$|\psi_0^{(3/2)}\rangle.$$

$(\mathcal{Q})^n |\psi_0^{(3/2)}\rangle$ denotes, symbolically, the set of states obtained by acting on the lowest weight state $|\psi_0^{(3/2)}\rangle$ n times with supersymmetry generators where $\mathcal{Q} = \{\mathfrak{Q}^r, \mathfrak{S}^r, \Pi^{mr}\}$ that determine the $SO^*(8)$ irreps and their $USp(2N)$ transformation properties uniquely.

Table 4: Below we give the minimal unitary supermultiplet of $\mathfrak{osp}(8^*|2N)$ defined by the lowest weight vector $|\psi_0^{(3/2)}\rangle$. The decomposition of lowest energy $SU(4)$ irreps of $SO^*(8)$ with respect to $SU(2)_T \times SU(2)_A \times U(1)_J$ subgroup is given in column 4. H is the AdS energy (negative conformal dimension), and \mathcal{H} is the total energy. The Dynkin labels of the lowest energy $SU(4)$ representations of $SO^*(8)$ coincide with the Dynkin labels of the corresponding massless $6D$ conformal fields with respect to the Lorentz group $SU^*(4)$. $USp(2N)$ Dynkin labels of these fields are also given.

State	H	\mathcal{H}	$(\mathfrak{t}, \mathfrak{a})^{\mathfrak{j}}$	$SU(4)$ Dynkin	$USp(2N)$ Dynkin
$ \Omega\rangle$	2	$2 - \frac{N}{2}$	$(0, 0)^0$	(0,0,0)	$\underbrace{(0, \dots, 0, 1)}_{(N-1)}$
$\mathcal{Q} \Omega\rangle$	$\frac{5}{2}$	$3 - \frac{N}{2}$	$(\frac{1}{2}, 0)^{-\frac{1}{2}} \oplus (0, \frac{1}{2})^{+\frac{1}{2}}$	(1,0,0)	$\underbrace{(0, \dots, 0, 1, 0)}_{(N-2)}$
$(\mathcal{Q})^2 \Omega\rangle$	3	$4 - \frac{N}{2}$	$(1, 0)^{-1} \oplus (\frac{1}{2}, \frac{1}{2})^0 \oplus (0, 1)^{+1}$	(2,0,0)	$\underbrace{(0, \dots, 0, 1, 0, 0)}_{(N-3)}$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$(\mathcal{Q})^n \Omega\rangle$	$2 + \frac{n}{2}$	$2 + n - \frac{N}{2}$	$(\frac{n}{2}, 0)^{-\frac{n}{2}} \oplus (\frac{n-1}{2}, \frac{1}{2})^{-\frac{n}{2}+1}$ $\oplus \dots$ $\oplus (\frac{1}{2}, \frac{n-1}{2})^{\frac{n}{2}-1} \oplus (0, \frac{n}{2})^{\frac{n}{2}}$	$(n, 0, 0)$	$\underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_{(N-n-1)} \underbrace{}_{(n)}$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$(\mathcal{Q})^N \Omega\rangle$	$2 + \frac{N}{2}$	$2 + \frac{N}{2}$	$(\frac{N}{2}, 0)^{-\frac{N}{2}} \oplus (\frac{N-1}{2}, \frac{1}{2})^{-\frac{N}{2}+1}$ $\oplus \dots$ $\oplus (\frac{1}{2}, \frac{N-1}{2})^{\frac{N}{2}-1} \oplus (0, \frac{N}{2})^{\frac{N}{2}}$	$(N, 0, 0)$	$(0, \dots, 0)$

11. Deformations of the Minimal Unitary Representation of $SO^*(8)$

Above we showed that the minrep of $SO^*(8)$ is simply the scalar doubleton representation that describes a conformal scalar field in six dimensions. The group $SO^*(8)$ admits infinitely many doubleton representations corresponding to $6D$ massless conformal fields of arbitrary spin [22, 59, 60]. They all can be constructed by the oscillator method over the Fock space of two pairs of twistorial oscillators transforming in the spinor representation of $SO^*(8)$. One would like to know whether the doubleton representations corresponding to massless conformal fields of higher spin can all be obtained from the minimal unitary representation by a “deformation” in a manner similar to what happens in the case of $4D$ conformal group $SU(2, 2)$ [55]. Remarkably, once again, we find that there exists infinitely many deformations of the minrep labeled by the spin (t) of an $SU(2)$ subgroup. By allowing this spin t to take on all possible values, we obtain all the doubleton irreps as deformations of the minrep of $SO^*(8)$.

To realize the deformations of the minimal representation of $SO^*(8)$, we first introduce an arbitrary number P pairs of fermionic oscillators ξ_x and χ_x and their hermitian conjugates $\xi^x = (\xi_x)^\dagger$ and $\chi^x = (\chi_x)^\dagger$ ($x = 1, 2, \dots, P$) that satisfy the usual anti-commutation relations

$$\{\xi_x, \xi^y\} = \{\chi_x, \chi^y\} = \delta_y^x \quad \{\xi_x, \xi_y\} = \{\xi_x, \chi_y\} = \{\chi_x, \chi_y\} = 0. \quad (11.1)$$

Note that the following bilinears of these fermionic oscillators

$$G_+ = \xi^x \chi_x \quad G_- = \chi^x \xi_x \quad G_0 = \frac{1}{2} (N_\xi - N_\chi) \quad (11.2)$$

where $N_\xi = \xi^x \xi_x$ and $N_\chi = \chi^x \chi_x$ are the respective number operators, generate an $\mathfrak{su}(2)_G$ algebra:

$$[G_+, G_-] = 2G_0 \quad [G_0, G_\pm] = \pm G_0 \quad (11.3)$$

We choose the Fock vacuum of these fermionic oscillators such that

$$\xi_x |0\rangle = \chi_x |0\rangle = 0 \quad (11.4)$$

for all $x = 1, 2, \dots, P$. Clearly a state of the form

$$\chi^{[1} \chi^2 \chi^3 \dots \chi^{P]} |0\rangle$$

has a definite eigenvalue of G_0 and is annihilated by the operator G_- . Note that square bracketing of fermionic indices imply complete anti-symmetrization of weight one. By acting on this state with the operator G_+ , one can obtain P other states, namely:

$$\xi^{[1} \chi^2 \chi^3 \dots \chi^{P]} |0\rangle \oplus \xi^{[1} \xi^2 \chi^3 \dots \chi^{P]} |0\rangle \oplus \dots \oplus \xi^{[1} \xi^2 \xi^3 \dots \xi^{P]} |0\rangle$$

This set of $P + 1$ states transforms irreducibly under $\mathfrak{su}(2)_G$ in the spin $t = \frac{P}{2}$ representation.

Recall that the “undeformed” minimal unitary realization of $\mathfrak{so}^*(8)$ has a 5-graded decomposition with respect to the subalgebra $\mathfrak{g}^{(0)} = \mathfrak{su}(2)_A \oplus \mathfrak{su}(1,1)_N \oplus \mathfrak{su}(2)_S \oplus \mathfrak{so}(1,1)$, as given in equation (4.26). Now to deform the minimal unitary realization of $\mathfrak{so}^*(8)$, we extend the subalgebra $\mathfrak{su}(2)_S$ to the diagonal subalgebra $\mathfrak{su}(2)_{\dot{T}}$ of $\mathfrak{su}(2)_S$ and $\mathfrak{su}(2)_G$. In other words, the generators of $\mathfrak{su}(2)_S$ receive contributions from the ξ - and χ -type fermionic oscillators as follows:

$$\begin{aligned}\dot{T}_+ &= S_+ + G_+ = a^m b_m + \xi^x \chi_x \\ \dot{T}_- &= S_- + G_- = b^m a_m + \chi^x \xi_x \\ \dot{T}_0 &= S_0 + G_0 = \frac{1}{2} (N_a - N_b + N_\xi - N_\chi)\end{aligned}\tag{11.5}$$

The quadratic Casimir of this subalgebra $\mathfrak{su}(2)_{\dot{T}}$ is given by

$$\mathcal{C}_2 \left[\mathfrak{su}(2)_{\dot{T}} \right] = \dot{\mathcal{T}}^2 = \dot{T}_0 \dot{T}_0 + \frac{1}{2} \left(\dot{T}_+ \dot{T}_- + \dot{T}_- \dot{T}_+ \right).\tag{11.6}$$

The 5-graded decomposition of the deformed minimal unitary realization, which we denote as $\mathfrak{so}^*(8)_D$, is now with respect to the subalgebra $\mathfrak{su}(2)_A \oplus \mathfrak{su}(1,1)_N \oplus \mathfrak{su}(2)_{\dot{T}} \oplus \mathfrak{so}(1,1)$, where, once again, the $\mathfrak{so}(1,1)$ generator Δ defines the 5-grading:

$$\mathfrak{so}^*(8)_D = \mathfrak{g}_D^{(-2)} \oplus \mathfrak{g}_D^{(-1)} \oplus \left[\mathfrak{su}(2)_A \oplus \mathfrak{su}(1,1)_N \oplus \mathfrak{su}(2)_{\dot{T}} \oplus \Delta \right] \oplus \mathfrak{g}_D^{(+1)} \oplus \mathfrak{g}_D^{(+2)}\tag{11.7}$$

The rest of the grade zero subspace, $\mathfrak{su}(2)_A \oplus \mathfrak{su}(1,1)_N \oplus \Delta$, remains unchanged under this deformation (see equation (4.10)). However, it should be noted that the quadratic Casimir of $\mathfrak{su}(2)_{\dot{T}}$ is no longer equal to those of $\mathfrak{su}(2)_A$ and $\mathfrak{su}(1,1)_N$.

Grade -2 and -1 generators of $\mathfrak{so}^*(8)_D$ are also the same as those of the undeformed $\mathfrak{so}^*(8)$:

$$\dot{K}_- = K_- = \frac{1}{2} x^2\tag{11.8}$$

$$\begin{aligned}\dot{U}_m &= U_m = x a_m & \dot{U}^m &= U^m = x a^m \\ \dot{V}_m &= V_m = x b_m & \dot{V}^m &= V^m = x b^m\end{aligned}\tag{11.9}$$

However, since $\mathfrak{su}(2)_S$ has now been extended to $\mathfrak{su}(2)_{\dot{T}}$, the grade $+2$ generator, which previously contained the quadratic Casimir \mathcal{S}^2 of $\mathfrak{su}(2)_S$, now depends on $\dot{\mathcal{T}}^2$:

$$\dot{K}_+ = \frac{1}{2} p^2 + \frac{1}{4 x^2} \left(8 \dot{\mathcal{T}}^2 + \frac{3}{2} \right).\tag{11.10}$$

The generators in grade $+1$ subspace are also modified since they are obtained from the commutators of the form $\left[\mathfrak{g}_D^{(-1)}, \mathfrak{g}_D^{(+2)} \right]$:

$$\begin{aligned}\dot{\tilde{U}}_m &= i \left[\dot{U}_m, \dot{K}_+ \right] & \dot{\tilde{U}}^m &= \left(\dot{\tilde{U}}_m \right)^\dagger = i \left[\dot{U}^m, \dot{K}_+ \right] \\ \dot{\tilde{V}}_m &= i \left[\dot{V}_m, \dot{K}_+ \right] & \dot{\tilde{V}}^m &= \left(\dot{\tilde{V}}_m \right)^\dagger = i \left[\dot{V}^m, \dot{K}_+ \right]\end{aligned}\tag{11.11}$$

The explicit form of these grade +1 generators are as follows:

$$\begin{aligned}
\mathring{\tilde{U}}_m &= -p a_m + \frac{2i}{x} \left[\left(\mathring{T}_0 + \frac{3}{4} \right) a_m + \mathring{T}_- b_m \right] \\
\mathring{\tilde{U}}^m &= -p a^m - \frac{2i}{x} \left[\left(\mathring{T}_0 - \frac{3}{4} \right) a^m + \mathring{T}_+ b^m \right] \\
\mathring{\tilde{V}}_m &= -p b_m - \frac{2i}{x} \left[\left(\mathring{T}_0 - \frac{3}{4} \right) b_m - \mathring{T}_+ a_m \right] \\
\mathring{\tilde{V}}^m &= -p b^m + \frac{2i}{x} \left[\left(\mathring{T}_0 + \frac{3}{4} \right) b^m - \mathring{T}_- a^m \right]
\end{aligned} \tag{11.12}$$

The deformed generators of $\mathfrak{so}^*(8)_D$ with “o” over them satisfy the same commutation relations as the corresponding “undeformed” generators of $\mathfrak{so}^*(8)$. Therefore, the 5-grading of $\mathfrak{so}^*(8)_D$, defined by Δ , takes the form:

$$\begin{aligned}
\mathfrak{so}^*(8)_D &= \mathbf{1} \oplus (\mathbf{4}, \mathbf{2}) \oplus \left[\mathfrak{su}(2)_A \oplus \mathfrak{su}(1, 1)_N \oplus \mathfrak{su}(2)_{\mathring{T}} \oplus \mathfrak{so}(1, 1)_\Delta \right] \oplus (\mathbf{4}, \mathbf{2}) \oplus \mathbf{1} \\
&= \mathring{K}_- \oplus \left[\mathring{U}_m, \mathring{U}^m, \mathring{V}_m, \mathring{V}^m \right] \oplus \left[A_{\pm, 0} \oplus N_{\pm, 0} \oplus \mathring{T}_{\pm, 0} \oplus \Delta \right] \\
&\quad \oplus \left[\mathring{\tilde{U}}_m, \mathring{\tilde{U}}^m, \mathring{\tilde{V}}_m, \mathring{\tilde{V}}^m \right] \oplus \mathring{K}_+
\end{aligned} \tag{11.13}$$

The quadratic Casimir of $\mathfrak{so}^*(8)_D$ is given by

$$\begin{aligned}
\mathcal{C}_2 [\mathfrak{so}^*(8)_D] &= \mathcal{C}_2 [\mathfrak{su}(2)_{\mathring{T}}] + \mathcal{C}_2 [\mathfrak{su}(2)_A] + \mathcal{C}_2 [\mathfrak{su}(1, 1)_N] + \mathcal{C}_2 [\mathfrak{su}(1, 1)_{\mathring{K}}] \\
&\quad - \frac{i}{4} \mathcal{F}(\mathring{U}, \mathring{V})
\end{aligned} \tag{11.14}$$

where

$$\begin{aligned}
\mathcal{F}(\mathring{U}, \mathring{V}) &= \left(\mathring{U}_m \mathring{\tilde{U}}^m + \mathring{V}_m \mathring{\tilde{V}}^m + \mathring{\tilde{U}}^m \mathring{U}_m + \mathring{\tilde{V}}^m \mathring{V}_m \right) \\
&\quad - \left(\mathring{U}^m \mathring{\tilde{U}}_m + \mathring{V}^m \mathring{\tilde{V}}_m + \mathring{\tilde{U}}_m \mathring{U}^m + \mathring{\tilde{V}}_m \mathring{V}^m \right)
\end{aligned} \tag{11.15}$$

and reduces to

$$\mathcal{C}_2 [\mathfrak{so}^*(8)_D] = 2 \mathcal{G}^2 - 4 \tag{11.16}$$

where \mathcal{G}^2 is the quadratic Casimir of $\mathfrak{su}(2)_G$.

11.1 The 3-grading of $SO^*(8)_D$ with respect to the subgroup $SU(4) \times U(1)$

The Lie algebra of $\mathfrak{so}^*(8)_D$ can be given a compact 3-grading

$$\mathfrak{so}^*(8)_D = \mathfrak{C}_D^- \oplus \mathfrak{C}_D^0 \oplus \mathfrak{C}_D^+ \tag{11.17}$$

with respect to its maximal compact subalgebra $\mathfrak{su}(4) \oplus \mathfrak{u}(1)$, determined by the $\mathfrak{u}(1)$ generator

$$\mathring{H} = N_0 + \frac{1}{2} \left(\mathring{K}_+ + \mathring{K}_- \right). \quad (11.18)$$

The generators that belong to the grade $0, \pm 1$ subspaces are as follows:

$$\begin{aligned} \mathfrak{E}_D^- &= \left(\mathring{U}_m - i \mathring{\tilde{U}}_m \right) \oplus \left(\mathring{V}_m - i \mathring{\tilde{V}}_m \right) \oplus N_- \oplus \left[\Delta + i \left(\mathring{K}_+ - \mathring{K}_- \right) \right] \\ \mathfrak{E}_D^0 &= \left[\mathring{T}_{\pm,0} \oplus A_{\pm,0} \oplus \left(N_0 - \frac{1}{2} \left(\mathring{K}_+ + \mathring{K}_- \right) \right) \right. \\ &\quad \left. \oplus \left(\mathring{U}_m + i \mathring{\tilde{U}}_m \right) \oplus \left(\mathring{V}_m + i \mathring{\tilde{V}}_m \right) \oplus \left(\mathring{U}^m - i \mathring{\tilde{U}}^m \right) \oplus \left(\mathring{V}^m - i \mathring{\tilde{V}}^m \right) \right] \oplus \mathring{H} \\ \mathfrak{E}_D^+ &= \left(\mathring{U}^m + i \mathring{\tilde{U}}^m \right) \oplus \left(\mathring{V}^m + i \mathring{\tilde{V}}^m \right) \oplus N_+ \oplus \left[\Delta - i \left(\mathring{K}_+ - \mathring{K}_- \right) \right] \end{aligned} \quad (11.19)$$

11.2 Deformed minreps of $SO^*(8)$ as massless $6D$ conformal fields

Consider the vacuum state $|0\rangle$ that is annihilated by the bosonic oscillators a_m, b_m ($m = 1, 2$) and the fermionic oscillators ξ_x, χ_x ($x = 1, 2, \dots, P$):

$$a_m |0\rangle = b_m |0\rangle = \xi_x |0\rangle = \chi_x |0\rangle = 0 \quad (11.20)$$

The tensor products of the states of the form $(a^m)^{n_{a,m}} |0\rangle, (b^m)^{n_{b,m}} |0\rangle, \xi^x |0\rangle$ and $\chi^x |0\rangle$, where $n_{a,m}$ and $n_{b,m}$ are non-negative integers, form a “particle basis” of states in this Fock space.

As the “particle basis” of the Hilbert space of the deformed minimal unitary representation of $SO^*(8)$, we take the tensor product of the above states with the state space of the singular (isotonic) oscillator:

$$(a^1)^{n_{a,1}} |0\rangle \otimes (a^2)^{n_{a,2}} |0\rangle \otimes (b^1)^{n_{b,1}} |0\rangle \otimes (b^2)^{n_{b,2}} |0\rangle \otimes \xi^{[x_1} \dots \xi^{x_k} \chi^{x_{k+1}} \dots \chi^{x_P]} |0\rangle \otimes \left| \psi_n^{(\alpha_t)} \right\rangle$$

where square brackets imply full antisymmetrization and denote them as

$$(a^1)^{n_{a,1}} (a^2)^{n_{a,2}} (b^1)^{n_{b,1}} (b^2)^{n_{b,2}} \xi^{[x_1} \dots \xi^{x_k} \chi^{x_{k+1}} \dots \chi^{x_P]} \left| \psi_n^{(\alpha_t)} \right\rangle$$

or simply as

$$\left| \psi_n^{(\alpha_t)} ; n_{a,1}, n_{a,2}, n_{b,1}, n_{b,2} ; \frac{P}{2}, k - \frac{P}{2} \right\rangle$$

where $k = 0, 1, 2, \dots, P$. Note that α_t now depends on the eigenvalue t of the quadratic Casimir of $SU(2)_{\hat{T}}$.

Note that the $(P+1)$ states

$$\left| \psi_n^{(\alpha_t)} ; 0, 0, 0, 0 ; \frac{P}{2}, k - \frac{P}{2} \right\rangle \quad k = 0, 1, \dots, P \quad (11.21)$$

are annihilated by all grade -1 operators in \mathfrak{C}_D^- and transforms in the spin $t = \frac{P}{2}$ representation of $\mathfrak{su}(2)_{\hat{T}}$, if α_t satisfies

$$\alpha_t = 2t + \frac{3}{2}. \quad (11.22)$$

These states have a definite eigenvalue of $t + 2$ with respect to \hat{H} (given in equation (11.18)):

$$\hat{H} \left| \psi_n^{(2t+3/2)} ; 0, 0, 0, 0 ; \frac{P}{2}, k - \frac{P}{2} \right\rangle = (t + 2) \left| \psi_n^{(2t+3/2)} ; 0, 0, 0, 0 ; \frac{P}{2}, k - \frac{P}{2} \right\rangle \quad (11.23)$$

By acting on these $(P + 1)$ states with the coset generators (C^{1m}, C^{2m}) of

$$SU(4) / \left[SU(2)_{\hat{T}} \times SU(2)_A \times U(1) \right]$$

one obtains a set of states, which we denote collectively as $|\Omega^{(2t+3/2)}\rangle$, transforming in an irrep of $SU(4)$ with Dynkin labels $(2t, 0, 0)$ that are eigenstates of \hat{H} with eigenvalue $(t + 2)$. The states $|\Omega^{(2t+3/2)}\rangle$ are annihilated by all the operators in \mathfrak{C}_D^- . Therefore, the deformed minimal unitary representation of $SO^*(8)$ is a unitary lowest weight representation. All the other states of the “particle basis” of the deformed minrep can be obtained from the set of states $|\Omega^{(2t+3/2)}\rangle$ by repeatedly acting on it with grade $+1$ operators in the \mathfrak{C}_D^+ subspace of $SO^*(8)_D$.

In Table 5, we present the deformed minrep of $SO^*(8)$. The notation $(\mathfrak{C}_D^+)^n |\Omega^{(\alpha_t)}\rangle$ represents all the states obtained by acting on the lowest weight state $|\Omega^{(\alpha_t)}\rangle$ with n grade $+1$ generators $(\hat{Y}^m, \hat{Z}^m, \hat{N}_+, \hat{B}_+)$. The deformed minrep with parameter t corresponds to a massless conformal field in six dimensions whose transformation under the $6D$ Lorentz group $SU^*(4)$ coincides with the transformation of the states $|\Omega^{(\alpha_t)}\rangle$ under the $SU(4)$ subgroup of $SO^*(8)_D$ and its conformal dimension is equal to $-(t + 2)$.

Table 5: $SU(4) \times U(1)$ decomposition of the deformed minimal unitary representation of $SO^*(8)$. For the deformed minrep with deformation parameter t , $\alpha_t = 2t + 3/2$. It corresponds to a massless $6D$ conformal field transforming in the $(2t, 0, 0)_{\text{Dynkin}}$ representation of the Lorentz group $SU^*(4)$ with conformal dimension $-(t + 2)$.

State	E	$SU(4)$ Dynkin
$ \Omega^{(\alpha_t)}\rangle$	$t + 2$	$(2t, 0, 0)$
$\mathfrak{C}_D^+ \Omega^{(\alpha_t)}\rangle$	$t + 3$	$(2t, 1, 0)$

Table 5: (continued)

State	E	$SU(4)$ Dynkin
$(\mathfrak{e}_D^+)^2 \Omega^{(\alpha_t)}\rangle$	$t + 4$	$(2t, 2, 0)$
\vdots	\vdots	\vdots
$(\mathfrak{e}_D^+)^n \Omega^{(\alpha_t)}\rangle$	$t + n + 2$	$(2t, n, 0)$
\vdots	\vdots	\vdots

12. Conclusions

In this paper we first studied the minimal unitary representation of $SO^*(8) \simeq SO(6, 2)$, which is the seven dimensional AdS group or the six dimensional conformal group, obtained by quantizing its quasiconformal realization. The resulting minrep coincides with the scalar doubleton representation of $SO^*(8)$, whose Poincaré limit in AdS_7 is singular.

We then introduced supersymmetry and extended these results to construct the minimal unitary supermultiplet of $OSp(8^*|2N)$, and, in particular, the minimal unitary supermultiplet of $OSp(8^*|4)$. The minimal unitary supermultiplet of $OSp(8^*|4)$ is simply the massless supermultiplet of $(2, 0)$ conformal field theory in six dimensions that is believed to be dual to M-theory on $AdS_7 \times S^4$.

Finally, we presented a method to introduce a family of deformations of the minrep of $SO^*(8)$, with respect to one of its $SU(2)$ subgroups. For each non-negative integer or half-integer value of the deformation parameter corresponding to the spin t of $SU(2)$ one obtains a unique positive energy unitary irreducible representation of $SO^*(8)$ which describes a massless conformal field of higher spin in six dimensions and coincide with the infinite family of doubletons studied in [22, 59, 60].

One can also obtain the “deformed” minimal unitary supermultiplets of $OSp(8^*|2N)$ by first deforming the minrep of $SO^*(8)$ and then extending it to the superalgebras $OSp(8^*|2N)$. These deformed minimal unitary supermultiplets correspond to six dimensional massless superconformal multiplets involving higher spin fields than the undeformed minimal unitary supermultiplet and will be given in a separate study [71].

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Appendix

A. The decomposition of $SO(6, 2)$ with respect to the subgroup $SO(4) \times SO(2, 2)$

The generators $\widetilde{M}_{ij} = i M_{ij}$ ($i, j, \dots = 1, 2, 3, 4$) form the $\mathfrak{so}(4)$ subalgebra of $\mathfrak{so}(6, 2) \supset \mathfrak{so}(4) \oplus \mathfrak{so}(2, 2)$. This $\mathfrak{so}(4)$ can be decomposed as a direct sum

$$\mathfrak{so}(4) = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \quad (\text{A.1})$$

where the generators of the two $\mathfrak{su}(2)$ subalgebras are given by:

$$\begin{aligned} L_1 &= \frac{1}{2} (\widetilde{M}_{23} - \widetilde{M}_{14}) & L_2 &= \frac{1}{2} (\widetilde{M}_{13} + \widetilde{M}_{24}) & L_3 &= \frac{1}{2} (\widetilde{M}_{12} - \widetilde{M}_{34}) \\ R_1 &= \frac{1}{2} (\widetilde{M}_{23} + \widetilde{M}_{14}) & R_2 &= \frac{1}{2} (\widetilde{M}_{13} - \widetilde{M}_{24}) & R_3 &= \frac{1}{2} (\widetilde{M}_{12} + \widetilde{M}_{34}) \end{aligned} \quad (\text{A.2})$$

They satisfy the commutation relations

$$\begin{aligned} [L_+, L_-] &= 2 L_3 & [L_3, L_\pm] &= \pm L_\pm \\ [R_+, R_-] &= 2 R_3 & [R_3, R_\pm] &= \pm R_\pm \end{aligned} \quad (\text{A.3})$$

where

$$\begin{aligned} L_\pm &= L_1 \pm i L_2 \\ R_\pm &= R_1 \pm i R_2. \end{aligned} \quad (\text{A.4})$$

The quadratic Casimir operators of the two $\mathfrak{su}(2)$'s are equal:

$$L^2 = L_1^2 + L_2^2 + L_3^2 = R^2 = R_1^2 + R_2^2 + R_3^2 = \frac{1}{8} \mathcal{I}_4 + 1 \quad (\text{A.5})$$

The centralizer of $SU(2)_L \times SU(2)_R$ within $SO(6, 2)$ is $SO(2, 2)$, which also decomposes as

$$SO(2, 2) = SU(1, 1)_{\mathfrak{L}} \times SU(1, 1)_{\mathfrak{R}} \quad (\text{A.6})$$

where $SU(1, 1)_{\mathfrak{L}}$ is generated by K_+, K_- and Δ and $SU(1, 1)_{\mathfrak{R}}$ is generated by J_\pm and J_0 .

In their compact bases the generators of $SU(1,1)_{\mathfrak{L}}$ and $SU(1,1)_{\mathfrak{R}}$ take the form

$$\begin{aligned}\mathfrak{L}_+ &= -\frac{1}{2} [\Delta - i(K_+ - K_-)] & \mathfrak{R}_+ &= -\frac{1}{2} \left[J_0 + \frac{i}{2} (J_+ - J_-) \right] \\ \mathfrak{L}_- &= -\frac{1}{2} [\Delta + i(K_+ - K_-)] & \mathfrak{R}_- &= -\frac{1}{2} \left[J_0 - \frac{i}{2} (J_+ - J_-) \right] \\ \mathfrak{L}_3 &= \frac{1}{2} (K_+ + K_-) & \mathfrak{R}_3 &= -\frac{1}{4} (J_+ + J_-)\end{aligned}\tag{A.7}$$

and satisfy the commutation relations:

$$\begin{aligned}[\mathfrak{L}_+, \mathfrak{L}_-] &= -2\mathfrak{L}_3 & [\mathfrak{L}_3, \mathfrak{L}_{\pm}] &= \pm \mathfrak{L}_{\pm} \\ [\mathfrak{R}_+, \mathfrak{R}_-] &= -2\mathfrak{R}_3 & [\mathfrak{R}_3, \mathfrak{R}_{\pm}] &= \pm \mathfrak{R}_{\pm}\end{aligned}\tag{A.8}$$

Their quadratic Casimir operators

$$\mathfrak{L}^2 = \mathfrak{L}_3^2 - \frac{1}{2} (\mathfrak{L}_+ \mathfrak{L}_- + \mathfrak{L}_- \mathfrak{L}_+) \quad \mathfrak{R}^2 = \mathfrak{R}_3^2 - \frac{1}{2} (\mathfrak{R}_+ \mathfrak{R}_- + \mathfrak{R}_- \mathfrak{R}_+)\tag{A.9}$$

coincide:

$$\mathfrak{L}^2 = \mathfrak{R}^2 = \frac{1}{8} \mathcal{I}_4 + 1\tag{A.10}$$

Thus the quadratic Casimirs of $SU(2)_L$, $SU(2)_R$, $SU(1,1)_{\mathfrak{L}}$ and $SU(1,1)_{\mathfrak{R}}$ are all equal within the minimal unitary realization of $SO(6,2)$.

B. The compact 3-grading of $SO^*(8)$ with respect to the subgroup $SU(4) \times U(1)$

The Lie algebra $\mathfrak{so}^*(8)$ can be given a compact 3-grading

$$\mathfrak{so}^*(8) = \mathfrak{C}^- \oplus \mathfrak{C}^0 \oplus \mathfrak{C}^+\tag{B.1}$$

with respect to its maximal compact subalgebra $\mathfrak{su}(4) \oplus \mathfrak{u}(1)$, determined by the $\mathfrak{u}(1)$ generator

$$H = N_0 + \frac{1}{2} (K_+ + K_-) .\tag{B.2}$$

The operators that belong to the grade $0, \pm 1$ subspaces are as follows:

$$\begin{aligned}\mathfrak{C}^- &= \left(U_m - i\tilde{U}_m \right) \oplus \left(V_m - i\tilde{V}_m \right) \oplus N_- \oplus [\Delta + i(K_+ - K_-)] \\ \mathfrak{C}^0 &= \left[S_{\pm,0} \oplus A_{\pm,0} \oplus \left(N_0 - \frac{1}{2} (K_+ + K_-) \right) \right. \\ &\quad \left. \oplus \left(U_m + i\tilde{U}_m \right) \oplus \left(V_m + i\tilde{V}_m \right) \oplus \left(U^m - i\tilde{U}^m \right) \oplus \left(V^m - i\tilde{V}^m \right) \right] \oplus H \\ \mathfrak{C}^+ &= \left(U^m + i\tilde{U}^m \right) \oplus \left(V^m + i\tilde{V}^m \right) \oplus N_+ \oplus [\Delta - i(K_+ - K_-)]\end{aligned}\tag{B.3}$$

It is convenient to express the generators of the $\mathfrak{su}(4)$ subalgebra in \mathfrak{C}^0 subspace in its $\mathfrak{su}(4) \supset \mathfrak{su}(2)_S \oplus \mathfrak{su}(2)_A \oplus \mathfrak{u}(1)_J$ decomposition, where

$$J = N_0 - \frac{1}{2}(K_+ + K_-) \quad (\text{B.4})$$

determines a 3-grading of $\mathfrak{su}(4)$. We note that $S_{\pm,0}$ and $A_{\pm,0}$ were given in equations (4.6) and (4.10). The remaining generators of $\mathfrak{su}(4)$ are given by

$$\begin{aligned} C_{1m} &= \frac{1}{2} (U_m + i \tilde{U}_m) & C_{2m} &= \frac{1}{2} (V_m + i \tilde{V}_m) \\ C^{1m} &= \frac{1}{2} (U^m - i \tilde{U}^m) & C^{2m} &= \frac{1}{2} (V^m - i \tilde{V}^m) . \end{aligned} \quad (\text{B.5})$$

Then the $\mathfrak{su}(4)$ algebra becomes

$$\begin{aligned} [S_{n'}^{m'}, S_{l'}^{k'}] &= \delta_{n'}^{k'} S_{l'}^{m'} - \delta_{l'}^{m'} S_{n'}^{k'} & [A_n^m, A_l^k] &= \delta_n^k A_l^m - \delta_l^m A_n^k \\ [C^{m'm}, C_{n'n}] &= \delta_n^m S_{n'}^{m'} + \delta_{n'}^{m'} A_n^m + \delta_{n'}^m \delta_n^{m'} J & \\ [S_{n'}^{m'}, C^{k'm}] &= \delta_{n'}^{k'} C^{m'm} - \frac{1}{2} \delta_{n'}^{m'} C^{k'm} & [A_n^m, C^{m'k}] &= \delta_n^k C^{m'm} - \frac{1}{2} \delta_n^m C^{m'k} . \end{aligned} \quad (\text{B.6})$$

where we have labeled the generators of $\mathfrak{su}(2)_S$ and $\mathfrak{su}(2)_A$ as S_n^m and A_n^m , respectively:

$$\begin{aligned} S_1^1 &= -S_2^2 = S_0 & S_1^2 &= S_+ & S_2^2 &= (S_1^2)^\dagger = S_- \\ A_1^1 &= -A_2^2 = A_0 & A_1^2 &= A_+ & A_2^2 &= (A_1^2)^\dagger = A_- \end{aligned} \quad (\text{B.7})$$

We shall label \mathfrak{C}^\pm operators as

$$\begin{aligned} Y_m &= \frac{1}{2} (U_m - i \tilde{U}_m) & Y^m &= \frac{1}{2} (U^m + i \tilde{U}^m) \\ Z_m &= \frac{1}{2} (V_m - i \tilde{V}_m) & Z^m &= \frac{1}{2} (V^m + i \tilde{V}^m) \\ N_- & & N_+ & \\ B_- &= \frac{i}{2} [\Delta + i(K_+ - K_-)] & B_+ &= -\frac{i}{2} [\Delta - i(K_+ - K_-)] . \end{aligned} \quad (\text{B.8})$$

The commutators $[\mathfrak{C}^-, \mathfrak{C}^+]$ close into \mathfrak{C}^0 :

$$\begin{aligned} [Y_m, Y^n] &= \delta_m^n H + \delta_m^n S_0 + A_m^n & [Z_m, Z^n] &= \delta_m^n H - \delta_m^n S_0 + A_m^n \\ [Y_m, Z^n] &= \delta_m^n S_- & [N_-, B_+] &= 0 \\ [Y_m, N_+] &= +\epsilon_{mn} C^{2n} & [Z_m, N_+] &= -\epsilon_{mn} C^{1n} \\ [Y_m, B_+] &= C_{1m} & [Z_m, B_+] &= C_{2m} \\ [N_-, N_+] &= H + J & [B_-, B_+] &= H - J \end{aligned} \quad (\text{B.9})$$

The quadratic Casimir of this subalgebra $\mathfrak{su}(4)$ is given by

$$\mathcal{C}_2[\mathfrak{su}(4)] = S_{n'}^{m'} S_{m'}^n + A_n^m A_m^n + (C^{m'm} C_{m'm} + C_{m'm} C^{m'm}) + J^2 . \quad (\text{B.10})$$

C. Transformations between $SO(6, 2)$ oscillators c_i and $SO^*(8)$ oscillators a_m, b_m

The minrep of $SO(6, 2)$ can be related to the minrep of $SO^*(8)$ very simply by rewriting the oscillators a_m, b_m and a^m, b^m in terms of c_i and c_i^\dagger as follows:

$$\begin{aligned}
a_1 &= -\frac{i}{\sqrt{2}}(c_1 + i c_2) & a^1 &= \frac{i}{\sqrt{2}}(c_1^\dagger - i c_2^\dagger) \\
a_2 &= \frac{1}{\sqrt{2}}(c_3 + i c_4) & a^2 &= \frac{1}{\sqrt{2}}(c_3^\dagger - i c_4^\dagger) \\
b_1 &= \frac{1}{\sqrt{2}}(c_3 - i c_4) & b^1 &= \frac{1}{\sqrt{2}}(c_3^\dagger + i c_4^\dagger) \\
b_2 &= -\frac{i}{\sqrt{2}}(c_1 - i c_2) & b^2 &= \frac{i}{\sqrt{2}}(c_1^\dagger + i c_2^\dagger)
\end{aligned} \tag{C.1}$$

Then it is easy to see that we have the following mapping between the subalgebra $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R \oplus \mathfrak{su}(1, 1)_\mathfrak{R}$ of $\mathfrak{so}(6, 2)$ and the subalgebra $\mathfrak{su}(2)_A \oplus \mathfrak{su}(2)_S \oplus \mathfrak{su}(1, 1)_N$ of $\mathfrak{so}^*(8)$:

$$\begin{aligned}
L_3 &\longrightarrow A_0 & R_3 &\longrightarrow S_0 & \mathfrak{R}_3 &\longrightarrow N_0 \\
L_+ &\longrightarrow i A_+ & R_+ &\longrightarrow i S_+ & \mathfrak{R}_+ &\longrightarrow -i N_+ \\
L_- &\longrightarrow -i A_- & R_- &\longrightarrow -i S_- & \mathfrak{R}_- &\longrightarrow i N_-
\end{aligned} \tag{C.2}$$

The relation between $\mathfrak{su}(1, 1)_\mathfrak{L}$ of $\mathfrak{so}(6, 2)$ and $\mathfrak{su}(1, 1)_K$ of $\mathfrak{so}^*(8)$ is quite straight forward.

D. The superalgebra $\mathfrak{osp}(8^*|2N)$ in the 5-grading with respect to the subsuperalgebra $\mathfrak{osp}(4^*|2N)$

We gave the explicit realization of the superalgebra $\mathfrak{osp}(8^*|2N)$ in the 5-grading with respect to the subsuperalgebra $\mathfrak{osp}(4^*|2N)$ in section 8. In this appendix, we provide the commutation relations between the generators of $\mathfrak{osp}(8^*|2N)$ in this basis.

The (super-)commutation relations between the generators of grade zero subspace $\mathfrak{g}^{(0)} = \mathfrak{osp}(4^*|2N)$ are as follows:

$$\begin{aligned}
\{\Pi_{mr}, \overline{\Pi}^{ns}\} &= \delta_r^s A_m^n - \delta_m^n M_r^s + \delta_r^s \delta_m^n N_0 & \{\Pi_{mr}, \Sigma_n^s\} &= \epsilon_{mn} \delta_r^s N_- \\
\{\Sigma_m^r, \overline{\Sigma}_s^n\} &= \delta_s^r A_m^n + \delta_m^n M_s^r + \delta_s^r \delta_m^n N_0 & \{\Pi_{mr}, \overline{\Sigma}_s^n\} &= -\delta_m^n S_{rs} \\
[A_n^m, \Pi_{kr}] &= -\delta_k^m \Pi_{nr} + \frac{1}{2} \delta_n^m \Pi_{kr} & [N_+, \Pi_{mr}] &= -\epsilon_{mn} \overline{\Sigma}_r^n \\
[A_n^m, \Sigma_k^r] &= -\delta_k^m \Sigma_n^r + \frac{1}{2} \delta_n^m \Sigma_k^r & [N_+, \Sigma_m^r] &= \epsilon_{mn} \overline{\Pi}^{nr} \\
[S_{rs}, \overline{\Pi}^{mt}] &= \delta_s^t \overline{\Sigma}_r^m + \delta_r^t \overline{\Sigma}_s^m & [N_-, \Pi_{mr}] &= 0 \\
[S_{rs}, \Sigma_m^t] &= -\delta_r^t \Pi_{ms} - \delta_s^t \Pi_{mr} & [N_-, \Sigma_m^r] &= 0 \\
[M_s^r, \Pi_{mt}] &= -\delta_t^r \Pi_{ms} & [N_0, \Pi_{mr}] &= -\frac{1}{2} \Pi_{mr} \\
[M_s^r, \Sigma_m^t] &= \delta_s^t \Sigma_m^r & [N_0, \Sigma_m^r] &= -\frac{1}{2} \Sigma_m^r \\
[S_{rs}, \Pi_{mt}] &= 0 & [S_{rs}, \overline{\Sigma}_t^m] &= 0
\end{aligned} \tag{D.1}$$

The anticommutators between the supersymmetry generators in $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ (given in equations (8.4) and (8.17)) close into the bosonic generators in $\mathfrak{g}^{(0)}$:

$$\begin{aligned}
\{Q_r, \tilde{Q}_s\} &= 0 & \{Q_r, \tilde{S}_s\} &= -2i S_{rs} \\
\{Q_r, \tilde{Q}^s\} &= -\delta_r^s \Delta - 2i \delta_r^s T_0 + 2i M_r^s & \{Q_r, \tilde{S}^s\} &= -2i \delta_r^s T_- \\
\{S_r, \tilde{S}_s\} &= 0 & \{S_r, \tilde{Q}_s\} &= +2i S_{rs} \\
\{S_r, \tilde{S}^s\} &= -\delta_r^s \Delta + 2i \delta_r^s T_0 + 2i M_r^s & \{S_r, \tilde{Q}^s\} &= -2i \delta_r^s T_+
\end{aligned} \tag{D.2}$$

Finally, the commutators between the bosonic (even) and fermionic (odd) generators of $\mathfrak{g}^{(-1)}$ and $\mathfrak{g}^{(+1)}$ subspaces close into the fermionic (odd) generators of $\mathfrak{g}^{(0)}$:

$$\begin{aligned}
[U_m, \tilde{Q}_r] &= 0 & [V_m, \tilde{Q}_r] &= +2i \Pi_{mr} \\
[U_m, \tilde{Q}^r] &= -2i \Sigma_m^r & [V_m, \tilde{Q}^r] &= 0 \\
[U_m, \tilde{S}_r] &= -2i \Pi_{mr} & [V_m, \tilde{S}_r] &= 0 \\
[U_m, \tilde{S}^r] &= 0 & [V_m, \tilde{S}^r] &= -2i \Sigma_m^r
\end{aligned} \tag{D.3}$$

$$\begin{aligned}
[Q_r, \tilde{U}_m] &= 0 & [Q_r, \tilde{V}_m] &= +2i \Pi_{mr} \\
[Q^r, \tilde{U}_m] &= -2i \Sigma_m^r & [Q^r, \tilde{V}_m] &= 0 \\
[S_r, \tilde{U}_m] &= -2i \Pi_{mr} & [S_r, \tilde{V}_m] &= 0 \\
[S^r, \tilde{U}_m] &= 0 & [S^r, \tilde{V}_m] &= -2i \Sigma_m^r
\end{aligned} \tag{D.4}$$

E. The superalgebra $\mathfrak{osp}(8^*|2N)$ in the 3-grading with respect to the sub-superalgebra $\mathfrak{u}(4|N)$

As shown in section 9, the superalgebra $\mathfrak{osp}(8^*|2N)$ has a 3-graded decomposition with respect to the sub-superalgebra $\mathfrak{u}(4|N)$. In this appendix we give the explicit form of the remaining bosonic and supersymmetry generators and some useful (super-)commutation relations among them.

The commutators $[\mathfrak{C}^-, \mathfrak{C}^+]$ close into \mathfrak{C}^0 :

$$\begin{aligned}
[Y_m, Y^n] &= \delta_m^n H_B + \delta_m^n T_0 + A_m^n & [Z_m, Z^n] &= \delta_m^n H_B - \delta_m^n T_0 + A_m^n \\
[Y_m, Z^n] &= \delta_m^n T_- & [N_-, B_+] &= 0 \\
[Y_m, N_+] &= +\epsilon_{mn} C^{2n} & [Z_m, N_+] &= -\epsilon_{mn} C^{1n} \\
[Y_m, B_+] &= C_{1m} & [Z_m, B_+] &= C_{2m} \\
[N_-, N_+] &= H_B + J & [B_-, B_+] &= H_B - J
\end{aligned} \tag{E.1}$$

where the generators C^{mn} and C_{mn} are coset generators $SU(4) / [SU(2)_T \times SU(2)_A \times U(1)_J]$ defined below.

The $\mathfrak{su}(4|N)$ part of \mathfrak{E}^0 has an even subalgebra $\mathfrak{su}(4|N) \supset \mathfrak{su}(4) \oplus \mathfrak{su}(N) \oplus \mathfrak{u}(1)_D$, where the $\mathfrak{u}(1)_D$ charge and $\mathfrak{su}(N)$ generators are given by

$$\begin{aligned} D &= \frac{1}{2} (K_+ + K_-) + \frac{2}{N} M_0 \\ \widetilde{M}_s^r &= \alpha^r \alpha_s - \beta_s \beta^r - \frac{2}{N} \delta_s^r M_0 \end{aligned} \quad (\text{E.2})$$

The generators of $\mathfrak{su}(4)$, in its $\mathfrak{su}(4) \supset \mathfrak{su}(2)_T \oplus \mathfrak{su}(2)_A \oplus \mathfrak{u}(1)_J$ decomposition, are realized as follows:

$$\begin{aligned} T_+ &= a^m b_m + \alpha^r \beta_r & A_+ &= a^1 a_2 + b^1 b_2 \\ T_- &= b^m a_m + \beta^r \alpha_r & A_- &= a_1 a^2 + b_1 b^2 \\ T_0 &= \frac{1}{2} (N_a - N_b + N_\alpha - N_\beta) & A_0 &= \frac{1}{2} (a^1 a_1 - a^2 a_2 + b^1 b_1 - b^2 b_2) \end{aligned} \quad (\text{E.3a})$$

$$\begin{aligned} J &= N_0 - \frac{1}{2} (K_+ + K_-) \\ &= -\frac{1}{4} (x^2 + p^2) - \frac{1}{8x^2} \left(8\mathcal{T}^2 + \frac{3}{2} \right) + \frac{1}{2} (N_a + N_b) + 1 \end{aligned} \quad (\text{E.3b})$$

$$\begin{aligned} C_{1m} &= \frac{1}{2} (U_m + i\widetilde{U}_m) = \frac{1}{2} (x - ip) a_m - \frac{1}{x} \left[\left(T_0 + \frac{3}{4} \right) a_m + T_- b_m \right] \\ C^{1m} &= \frac{1}{2} (U^m - i\widetilde{U}^m) = \frac{1}{2} (x + ip) a^m - \frac{1}{x} \left[\left(T_0 - \frac{3}{4} \right) a^m + T_+ b^m \right] \\ C_{2m} &= \frac{1}{2} (V_m + i\widetilde{V}_m) = \frac{1}{2} (x - ip) b_m + \frac{1}{x} \left[\left(T_0 - \frac{3}{4} \right) b_m - T_+ a_m \right] \\ C^{2m} &= \frac{1}{2} (V^m - i\widetilde{V}^m) = \frac{1}{2} (x + ip) b^m + \frac{1}{x} \left[\left(T_0 + \frac{3}{4} \right) b^m - T_- a^m \right] \end{aligned} \quad (\text{E.3c})$$

One half of total supersymmetry generators of $\mathfrak{osp}(8^*|2N)$ belong to grade zero space, as part of the subsuperalgebra $\mathfrak{su}(4|N)$. Below we list these $8N$ supersymmetry generators:

$$\begin{aligned} \widetilde{\mathfrak{Q}}_r &= \frac{1}{2} (Q_r + i\widetilde{Q}_r) = \frac{1}{2} (x - ip) \alpha_r - \frac{1}{x} \left[\left(T_0 + \frac{3}{4} \right) \alpha_r + T_- \beta_r \right] \\ \widetilde{\mathfrak{Q}}^r &= \frac{1}{2} (Q^r - i\widetilde{Q}^r) = \frac{1}{2} (x + ip) \alpha^r - \frac{1}{x} \left[\left(T_0 - \frac{3}{4} \right) \alpha^r + T_+ \beta^r \right] \\ \widetilde{\mathfrak{S}}_r &= \frac{1}{2} (S_r + i\widetilde{S}_r) = \frac{1}{2} (x - ip) \beta_r + \frac{1}{x} \left[\left(T_0 - \frac{3}{4} \right) \beta_r - T_+ \alpha_r \right] \\ \widetilde{\mathfrak{S}}^r &= \frac{1}{2} (S^r - i\widetilde{S}^r) = \frac{1}{2} (x + ip) \beta^r + \frac{1}{x} \left[\left(T_0 + \frac{3}{4} \right) \beta^r - T_- \alpha^r \right] \\ \widetilde{\Sigma}_m^r &= \Sigma_m^r = a_m \alpha^r + b_m \beta^r \\ \widetilde{\Sigma}_r^m &= \overline{\Sigma}_r^m = a^m \alpha_r + b^m \beta_r \end{aligned} \quad (\text{E.4})$$

Under supercommutation, they close in to the bosonic generators of $\mathfrak{su}(4|N)$:

$$\begin{aligned}
\left\{ \tilde{\mathfrak{Q}}_r, \tilde{\mathfrak{Q}}^s \right\} &= -\delta_r^s T_0 + \tilde{M}_r^s + \delta_r^s D & \left\{ \tilde{\mathfrak{Q}}_r, \tilde{\mathfrak{S}}^s \right\} &= -\delta_r^s T_- \\
\left\{ \tilde{\mathfrak{S}}_r, \tilde{\mathfrak{S}}^s \right\} &= +\delta_r^s T_0 + \tilde{M}_r^s + \delta_r^s D & \left\{ \tilde{\Sigma}_m^r, \tilde{\mathfrak{Q}}_s \right\} &= \delta_s^r C_{1m} \\
\left\{ \tilde{\Sigma}_m^r, \tilde{\Sigma}_s^n \right\} &= \delta_s^r A_m^n + \delta_m^n \tilde{M}_s^r + \delta_s^r \delta_m^n D + \delta_s^r \delta_m^n J & \left\{ \tilde{\Sigma}_m^r, \tilde{\mathfrak{S}}_s \right\} &= \delta_s^r C_{2m}
\end{aligned} \tag{E.5}$$

These supersymmetry generators in \mathfrak{C}^\pm satisfy the following (anti-)commutation relations:

$$\begin{aligned}
\left\{ \mathfrak{Q}_r, \mathfrak{Q}^s \right\} &= +\delta_r^s T_0 - \tilde{M}_r^s + \delta_r^s H_\odot - \frac{2}{N} \delta_r^s M_0 \\
\left\{ \mathfrak{S}_r, \mathfrak{S}^s \right\} &= -\delta_r^s T_0 - \tilde{M}_r^s + \delta_r^s H_\odot - \frac{2}{N} \delta_r^s M_0 \\
\left\{ \Pi_{mr}, \bar{\Pi}^{ns} \right\} &= \delta_r^s A_m^n - \delta_m^n \tilde{M}_r^s + \delta_m^n \delta_r^s N_0 - \frac{2}{N} \delta_m^n \delta_r^s M_0 \\
\left\{ \mathfrak{Q}_r, \mathfrak{S}^s \right\} &= \delta_r^s T_- \\
\left\{ \Pi_{mr}, \mathfrak{Q}^s \right\} &= -\delta_r^s C_{2m} \\
\left\{ \Pi_{mr}, \mathfrak{S}^s \right\} &= +\delta_r^s C_{1m}
\end{aligned} \tag{E.6}$$

The commutators between bosonic operators in \mathfrak{C}^- and supersymmetry generators in \mathfrak{C}^+ are as follows:

$$\begin{aligned}
[Y_n, \mathfrak{Q}^r] &= \tilde{\Sigma}_n^r & [Y_n, \mathfrak{S}^r] &= 0 & [Y_n, \bar{\Pi}^{mr}] &= +\delta_n^m \tilde{\mathfrak{S}}^r \\
[Z_n, \mathfrak{Q}^r] &= 0 & [Z_n, \mathfrak{S}^r] &= \tilde{\Sigma}_n^r & [Z_n, \bar{\Pi}^{mr}] &= -\delta_n^m \tilde{\mathfrak{Q}}^r \\
[N_-, \mathfrak{Q}^r] &= 0 & [N_-, \mathfrak{S}^r] &= 0 & [N_-, \bar{\Pi}^{mr}] &= \epsilon^{mn} \tilde{\Sigma}_n^r \\
[B_-, \mathfrak{Q}^r] &= \tilde{\mathfrak{Q}}^r & [B_-, \mathfrak{S}^r] &= \tilde{\mathfrak{S}}^r & [B_-, \bar{\Pi}^{mr}] &= 0 \\
[S_{st}, \mathfrak{Q}^r] &= -2 \delta_{(s}^r \tilde{\mathfrak{S}}_{t)} & [S_{st}, \mathfrak{S}^r] &= +2 \delta_{(s}^r \tilde{\mathfrak{Q}}_{t)} & [S_{st}, \bar{\Pi}^{mr}] &= 2 \delta_{(s}^r \tilde{\Sigma}_{t)}^m
\end{aligned} \tag{E.7}$$

Note that we have used the notation “(st)” to indicate symmetrization of indices s and t with weight 1.

The anticommutators of supersymmetry generators in \mathfrak{C}^0 and those in \mathfrak{C}^+ can be written as

$$\begin{aligned}
\left\{ \tilde{\mathfrak{Q}}_r, \mathfrak{Q}^s \right\} &= \delta_r^s B_+ & \left\{ \tilde{\mathfrak{Q}}_r, \mathfrak{S}^s \right\} &= 0 & \left\{ \tilde{\mathfrak{Q}}_r, \bar{\Pi}^{ns} \right\} &= -\delta_r^s Z^n \\
\left\{ \tilde{\mathfrak{Q}}^r, \mathfrak{Q}^s \right\} &= 0 & \left\{ \tilde{\mathfrak{Q}}^r, \mathfrak{S}^s \right\} &= -S^{rs} & \left\{ \tilde{\mathfrak{Q}}^r, \bar{\Pi}^{ns} \right\} &= 0 \\
\left\{ \tilde{\mathfrak{S}}_r, \mathfrak{Q}^s \right\} &= 0 & \left\{ \tilde{\mathfrak{S}}_r, \mathfrak{S}^s \right\} &= \delta_r^s B_+ & \left\{ \tilde{\mathfrak{S}}_r, \bar{\Pi}^{ns} \right\} &= +\delta_r^s Y^n \\
\left\{ \tilde{\mathfrak{S}}^r, \mathfrak{Q}^s \right\} &= +S^{rs} & \left\{ \tilde{\mathfrak{S}}^r, \mathfrak{S}^s \right\} &= 0 & \left\{ \tilde{\mathfrak{S}}^r, \bar{\Pi}^{ns} \right\} &= 0 \\
\left\{ \tilde{\Sigma}_r^m, \mathfrak{Q}^s \right\} &= +\delta_r^s Y^m & \left\{ \tilde{\Sigma}_r^m, \mathfrak{S}^s \right\} &= +\delta_r^s Z^m & \left\{ \tilde{\Sigma}_r^m, \bar{\Pi}^{ns} \right\} &= -\epsilon^{mn} \delta_r^s N_+ \\
\left\{ \tilde{\Sigma}_m^r, \mathfrak{Q}^s \right\} &= 0 & \left\{ \tilde{\Sigma}_m^r, \mathfrak{S}^s \right\} &= 0 & \left\{ \tilde{\Sigma}_m^r, \bar{\Pi}^{ns} \right\} &= -\delta_m^n S^{rs}
\end{aligned} \tag{E.8}$$

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